On Aggregate Fluctuations, Systemic Risk, and the Covariance of Firm-Level Activity

Rory Mullen*
Warwick Business School

September 4, 2020

Abstract

Pairwise covariances of firm growth rates appear to drive the variance of aggregate growth rates in productivity, sales, and profit for public firms in the United States over the last half-century. High-productivity firms contribute most to the covariances driving aggregate variance, but least per dollar of market value that they generate. This fact may explain why investors demand lower returns from high-productivity firms. A tractable model of within-firm diversification qualitatively matches the empirical evidence, generating endogenous first and second moments of firm and aggregate productivity, and endogenous comovement between firm and aggregate productivity. In the model, movements in firm productivity drive movements in firm sales and profit, and firms' expected excess stock returns rise as firms' productivities covary more with aggregate productivity, relative to their market values. A regression analysis lends tentative empirical support to several predictions of the model.

*Email: rory.mullen@wbs.ac.uk. I thank my dissertation committee Fabio Ghironi (chair), Yu-chin Chen, and Philip Brock for their guidance. I thank University of Washington workshop and seminar participants for valuable comments. All remaining errors are my own.

1 Introduction

I drive to work—it's faster, but driving is risky. My point is this: some risks I choose, they aren't imposed upon me. Firms also choose risks: they choose their production methods, they choose their product lines, and these choices entail technology- and product-specific risks. When many firms choose similar risks, perhaps being drawn to similar rewards, their fortunes then rise or fall together. This comovement of firms' fortunes creates aggregate fluctuations, and the risks firms choose become systemic. To motivate the mechanism I have just described, I document four facts related to the comovement of firm-level activity for a large panel of public firms in the United States over the last half-century, and develop a model economy that matches the facts endogenously by letting firms choose to take some risks and avoid others. My contributions build on recent work on endogenous uncertainty in macroeconomic models, on the microeconomic origins of aggregate fluctuations, and on risks to equity owners in production economies.

The motivating evidence that firm-level covariance may drive aggregate fluctuations comes from a well-known decomposition: aggregate variance equals the sum of individual variances and pairwise covariances. Details of the decomposition depend on the aggregate in question—whether its elements are additively separable, whether they are growth rates or levels, whether entry and exit occur—but often a simple mathematical identity or approximation holds, and is useful for thinking about the sources of aggregate variance. In the simplest version, where x_ω is a firm variable in levels and $X = \sum_\Omega x_\omega$ the corresponding aggregate, and where ω, ω' are firms in Ω,

$$\begin{aligned}\mathrm{Var}(X) &= \mathrm{Var}\left(\sum_{\Omega} x_\omega\right) \\ &= \sum_{\Omega} \mathrm{Var}(x_\omega) + \sum_{\Omega} \sum_{\Omega \setminus \{\omega\}} \mathrm{Cov}(x_\omega, x_{\omega'}) \\ &= \sum_{\Omega} \mathrm{Cov}(x_\omega, X).\end{aligned} \tag{1}$$

The second line in equation (1) gives the above-mentioned decomposition into individual variances and pairwise covariances. The third line in equation (1) is also useful: it says that a firm's covariance *with* the aggregate is simultaneously its contribution *to* aggregate variance.

Now consider productivity, sales, and profit growth for Compustat firms: on average over the last half-century, covariances between firm growth rates for a given variable (productivity, sales, or profit) accounted for at between 80% and 90% of the variance of the aggregate growth rate for that variable. The median firm in the high-productivity decile contributed over 13 times as much variance to aggregate growth rates as the median firm in the full sample, and most of this contribution came from covariance with other firms. High-productivity firms drive aggregate fluctuations in productivity, sales, and profit growth rates, and do so through covariance with other firms. There are about 7,500 distinct firms in the Compustat sample over this period, coming from nearly all industries, and producing goods equal in value to about 20% of U.S. gross domestic product each year. Covariance matters for the aggregate fluctuations of these firms.

It also matters for their risk. Markowitz made this point in 1952 for portfolio returns, using a version of the above decomposition with weights. Covariance risk also underlies the portfolio-based capital asset pricing models of Sharpe (1964), Lintner (1965), and Mossin

(1966), and also Breeden (1979)'s consumption-based model. Mine is a productivity-based model in which a firm's risk depends, predominantly, on the covariance between the firm's productivity and aggregate productivity divided by the firm's market value—the last term on the right-hand side my model's equation for expected excess returns:

$$\mathrm{E}_t\left[r_{t+1}(\omega) - r_{f,t+1}\right] \approx \zeta_{r1}\frac{\mu(\omega)}{V_t(\omega)} + \zeta_{r2}\frac{\sigma_{\omega\Omega}(\omega)}{V_t(\omega)}, \tag{2}$$

where $V_t(\omega)$ is the firm's market value, $\mu(\omega)$ its expected productivity, and $\sigma_{\omega\Omega}(\omega)$ its covariance with aggregate productivity. This ratio, firm-aggregate productivity covariance over market value, may help explain why low-productivity firms pay investors significantly higher returns than high-productivity firms, as İmrohoroğlu and Tuzel (2014) have recently pointed out, despite high-productivity firms driving aggregate fluctuations. Evidence from Compustat suggests that low-productivity firms expose investors to more covariance risk per dollar invested in the firm. While the firm-aggregate covariances of the median high-productivity firm is thirteen times that of the full-sample median firm, this multiple falls to seventy percent, on a dollar-for-dollar basis, after dividing by market value, as figure 1 illustrates.

But why is firm aggregate productivity covariance over market value lower for high-productivity firms? Evidently, high-productivity firms are doing at least some business that investors value, but that weakly covaries with the business of other firms. I show in Section 4 that the cross-sectional evidence on this ratio can be explained by business-line diversification. Empirically, high-productivity firms do operate more business lines: 2.6 on average, against 1.5 for low-productivity firms (see table 1). Perhaps the additional segments at high-productivity firms yield diversification benefits. In theory, risk-averse investors would accept lower returns from high-productivity firms, if activities in some of the extra segments covaried less with aggregate activity, relative to the profit investors expected.[1] I investigate this explanation theoretically using a model of firm-level diversification in section 3.

Figure 1 summarizes the empirical evidence on firm-level covariance in productivity, sales, and profit growth that motivates my theory. To highlight the pervasiveness of covariance in all stages of value creation, the figure illustrates the evidence for growth rates of three variables.[2] Panel (a) shows the fraction of variance in aggregate growth rates due to pairwise covariance in firm-level growth rates. On average, pairwise covariances account for about 85% of aggregate variance each year. Panel (b) illustrates firm-aggregate covariance for firms sorted into productivity deciles. Recall that firm-aggregate covariance measures an individual firm's contribution to aggregate variance; high-productivity firms contribute far more than the median firm contributes. Panel (c) illustrates firm-aggregate covariance relative to market value. Interpret this statistic as a metric for the risk that firms poses to investors per dollar invested; by this metric, high-productivity firms expose investors much less risk than the median firm. This firm-level evidence on covariance provides a useful yardstick to measure the success of economic models explaining cross-sectional stock returns, or the success of economic models in explaining the microeconomic origins of aggregate fluctuations. My first contribution is to document this firm-level evidence.

Because firm-level productivity covariance appears to drive both firm risk and aggregate fluctuations in the data, a single model capable of producing an empirically plausible firm-level covariance structure would contribute to explanations of both firm risk and aggregate fluctuations. My second contribution is along these lines. I construct a DSGE model in

[1] Authors in the finance literature have primarily focused on the impact of firm diversification on firm value. Villalonga (2004) finds a value premium on diversified firms, consistent with the theory here. Here, the emphasis is on diversification and stock returns. In this context, Wang (2012) finds that less-diversified firms pay higher returns, consistent with the theory presented here. I discuss this further in section 2.

[2] Through the exposition, productivity refers to measured revenue total factor productivity, and in the empirical applications is estimated following Olley and Pakes (1996) and İmrohoroğlu and Tuzel (2014). Sales refers to net sales; profit refers to operating income before depreciation, both as reported in 10-K financial statements filed with the SEC, unless adjusted by Compustat.

which firm-level productivity covariance arises endogenously. The model produces aggregate fluctuations and cross-sectional patterns in firm-level systematic risk that are consistent with the empirical patterns. I endogenize covariance by allowing heterogeneous firms to choose risky business lines for themselves, from a menu that I specify exogenously. When firms choose similar business lines, their productivities covary. Mathematically, covariance in the model has a simple factor structure that is flexible and tractable. The mechanism relies on high-productivity firms choosing to operate a greater number of business lines, just as firms do in the Compustat data. My model predicts that firms with higher productivity have higher firm-aggregate productivity covariance, but lower firm-aggregate productivity covariance per dollar of market value. The motivating evidence in Figure 1 demonstrates the empirical plausibility of these model predictions, in particular for productivity and sales. Regressions controlling for alternative explanations provide formal inference. For productivity, sales, and profit, I regress firm-aggregate covariances of productivity, sales, and profit growth rates on firm-level productivity and a set of control variables, and find support for two model predictions. The model also predicts that firms with similar productivities have higher correlations with each other, and this pattern is weakly visible in the data and illustrated in panel (d) of Figure 1.

The rest of the paper is organized as follows: Section 2 relates this work to existing literature. Section 3 introduces a formal model based on business-line diversification. Section 4 presents propositions that describe the model's main mechanism, and demonstrate the model's qualitative consistency with the motivating empirical evidence. Section 5 provides details on the productivity estimation procedure and firm-level covariance calculations used to produce the motivating evidence in Figure 1. Section 5 also presents regressions that check the empirical plausibility of the business-line diversification hypothesis. Section 6 concludes.

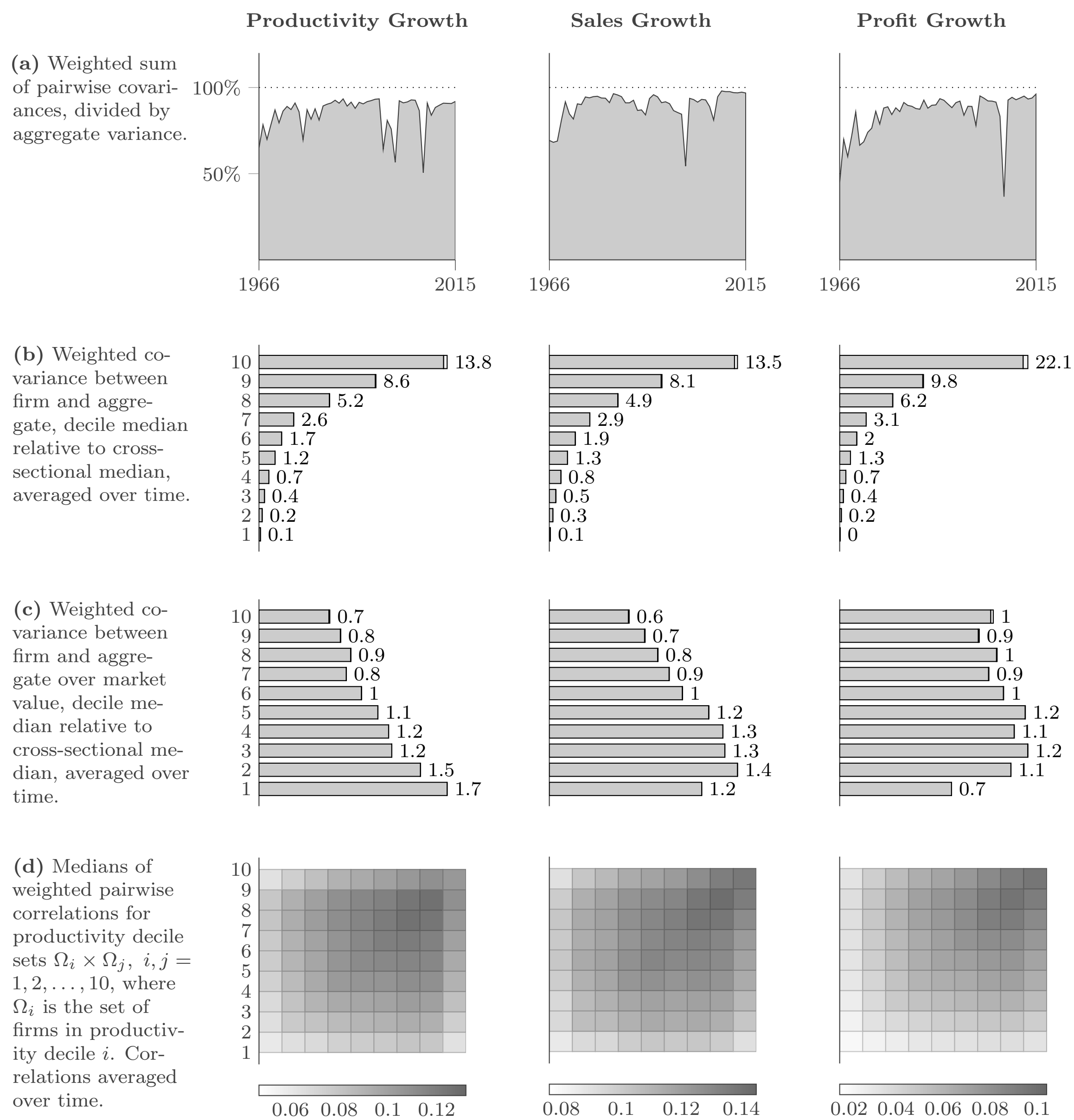


Figure 1 – **Evidence on Compustat Firm-Level Covariance Structure.** Compustat Annual Fundamentals, 1966-2015. Firm-level covariance statistics computed in 6-year rolling windows; firms grouped into productivity deciles using moving averages of estimated total factor productivities. Productivity estimates are free of time-industry effects. Sample includes all Compustat firms with no missing values in each 6-year window, excluding financial and utilities firms, firms with large merges, and the smallest 10% of firms by market value. Details are given in section 5.

2 Literature

This section relates my work to recent work on endogenous uncertainty in macroeconomic models, microeconomic origins of aggregate fluctuations, financial risk in production economies, business cycles in the firm cross section, and corporate diversification. A notable feature of the model I propose is that it accords, at least qualitatively, with several disparate areas of economic research. For example, the model in Section 3 endogenously generates aggregate uncertainty from microeconomic shocks, captures features of the cross-section of stock returns, matches a host of stylized facts related to cross-sectional volatility and co-movement, and captures empirical regularities related to the number of technologies, product lines, and business segments firms operate. The following paragraphs discuss connections between the this work and several related areas of research.

Endogenous uncertainty in macroeconomic models. The financial crisis of 2007 led to calls for macroeconomic models in which aggregate fluctuations arise endogenously. Romer (2016) complains about models in which aggregate fluctuations "are not influenced by the action that any person takes." Stiglitz (2011) questions the relevance of real business cycle models for work on recessions, because the models presume "that the origin of fluctuations [is] exogenous." These criticisms apply only partially to the class of models to which mine belongs: production economies with firm heterogeneity and random productivity. In many models within this class, exogenous productivity shocks at the firm level propagate endogenously from firms to aggregate variables. Because the propagation is endogenous, the aggregate fluctuations are endogenous. Dutta and Polemarchakis (1992) offer an early example of this type of propagation, Gabaix (2011) and Acemoglu et al. (2012) are recent example, and I review more of this literature below.

Still, within this class of models, productivity shocks are often exogenous at the firm level, so at the firm level the Stiglitz-Romer critique is valid in many cases. Decker et al. (2016)

provide a notable exception. In their framework, firms choose markets in which to operate, and market exposure determines firm risk. In contrast to Decker et al., who study changes in idiosyncratic firm risk and business cycles, I study systematic firm risk and cross-sectional stock returns. In my model, systematic risk arises endogenously: individual firms choose their production technologies, and their choices determine the probability distribution of their productivity shocks. When many firms choose the same technology, shocks to that technology generate comovement amongst firms using it, and the comovement generates aggregate fluctuations. In this sense, I provide a framework for addressing the Stiglitz-Romer critique in production economies with heterogeneous firms and random productivity.

Microeconomic origins of aggregate fluctuations.

The literature on aggregate uncertainty arising from individual uncertainty—what has been called the microeconomic origins of aggregate fluctuations— was born out of a literature on aggregate **certainty** arising from individual uncertainty. I think I can cite an early paper coauthored by French, and maybe some papers mentioned in **Kirman:1981aa** and then, for a more recent example, in general equilibrium theory, cite Malinvaud (1972), as he considers a large endowment economy and finds conditions under which equilibrium prices are "sure" despite agents facing individual uncertainty. Feldman and Gilles (1985) comment on this topic somehow.

In a first wave of research, economists then turned to the opposite case of aggregate uncertainty arising from individual uncertainty. **Jovanovic:1987aa** and **Bak:1993aa** are good examples of this. Malinvaud (1972) in fact also belongs to this wave, because he briefly considers the case where cross-sectionally correlated individual uncertainty gives rise to aggregate uncertainty, but it isn't his "first aim." Al-Najjar (1995) does primarily focus on generating aggregate uncertainty from individual uncertainty, and provides a more general construction for economies with a continuum of agents. While both Malinvaud and

Al-Najjar (1995) impose the cross-sectional correlation structure exogenously, the model I propose gives rise to it endogenously in a general equilibrium production economy.[3]

although his purpose was not to generate aggregate uncertainty specifically, Hulten (1978) may also belong to this first wave, if only because it motivated a second wave. Hulten showed how changes in aggregate output and productivity can be expressed as a weighted average of changes in individual output and productivity. Gabaix (2011) then picked this insight up showed how aggregate uncertainty can arise even from idiosyncratic individual uncertainty, as long as the aggregation weights are sufficiently skewed in their distribution across agents. Carvalho and Gabaix (2013) use a granular model economy to explain movements in U.S. GDP volatility. Acemoglu et al. (2012) show how input-output networks can magnify idiosyncratic shocks, focusing on sectoral shocks, and Herskovic et al. (2017) develop a model in which granular effects at the firm level lead to sectoral shocks that propagate through networks and generate aggregate fluctuations. Baqaee and Farhi (2019) find that idiosyncratic shocks have larger aggregate effects when Hulten's approximation is taken to a second order.

Where the above papers have all focused on economies with a countable number of agents, there is a parallel literature that focuses on economies with a continuum of agents. These are sometimes referred to as countable and continuum economies. In continuum economies, only Malinvaud (1972) considers how "collective risks", as he calls them, can be generated from "individual risks" faced by individual agents. He finds that cross-sectional correlation, —that is, single risks that commonly affect all agents in an economy. Malinvaud shows how collective risks can arise in continuum economies where atomless agents face individual but not independent risks. Collective risks arise through cross-sectional correlations in the individual risks.

[3] **Kirman:1981aa** gives a nice history of continuum economies and goes over some of the measure theory involved in handling them rigorously.

Specifically in the context of aggregate fluctuations, there is In the More recently,

My work differs from the recent work because I emphasize firm-level shocks that covary, rather than independent firm-level shocks, as a crucial firm-level driver of aggregate fluctuations. My focus on covariance is motivated by the Compustat evidence, where covariance in firm-level activity appears to account for between 80% and 90% of variance in aggregate activity—and this applies also to productivity growth rates, which are often modeled in the literature as idiosyncratic. While none of the recent work places quite the same emphasis on covariance, Baqaee and Farhi (2019) do consider the case of correlated shocks in their second-order approximations and find that correlated shocks can have large aggregate effects. Malinvaud (1972) considers a sequence of Dutta and Polemarchakis (1992) briefly consider cases where microeconomic shocks covary locally, but where covariance decays rapidly enough in the distance between the shocks that no fluctuations are generated. In my model, covariance does not decay rapidly: roughly, this is because some technologies are used by nearly all firms and are therefore highly systemic. This statement is true even for a continuum of technologies and a continuum of firms. I provide formal justification in the propositions in Section 4.

Financial risk in production economies. Explaining cross-sectional differences in stock returns has long been of interest in finance. In a 2011 survey covering thirty years of research on cross-sectional differences in stock returns, van Dijk remarks on a trend toward asset pricing in general equilibrium production economies. This trending line of research seeks an "economic theory that identifies the state variables that drive variation in returns related to firm size."

Berk et al. (1999) provide the prototypical example, albeit in a partial equilibrium setting, where firms invest in projects with uncertain cash flows and durations. The cash flows covary with an exogenous pricing kernel, and a firm's set of active projects determines the firm's risk. My work is closely related to Berk et al., with production technologies here playing a similar role to investment projects there. Our models differ in two important ways: first, in my work each technology's covariance with aggregate productivity is endogenous—it depends on the number of firms that choose to operate the technology. Second, my pricing kernel is endogenous, and, through market clearing conditions, ultimately also depends on firm-level technology choices.

Gomes (2001) develops a general equilibrium model in which financing constraints generate an empirically plausible cross section of stock returns. More recent models have relied on convex capital adjustment costs to produce differences in returns; examples include Gomes et al. (2003), İmrohoroğlu and Tuzel (2014), and Zhang (2017). However, Clementi and Palazzo (2018) find that capital adjustment costs are empirically too small to explain cross-sectional differences in returns, and suggest that additional explanations are needed. Donangelo et al. (2017) propose a novel labor leverage mechanism. I view my work as complementary to these lines. I emphasize business line diversification as a source of cross-sectional differences in returns. While the diversification argument is not new to the finance literature, my implementation in a general equilibrium production economy is new, to the best of my knowledge. I discuss the diversification literature further below.

Finally, the real business cycle literature also studies stock returns in production economies, but focuses mostly on the time series properties of returns. Examples include Rouwenhorst (1995), Jermann (1998), and Lettau (2003). In my work, as in the traditional real business cycle literature, fluctuations are driven by technology shocks. But in my work there is a continuum of technologies and firms solve a technology choice problem; these features provide a microeconomic foundation for the traditional aggregate technology shock, and, in particular, one that is consistent with the cross-sectional evidence on stock returns.

The cross section of firms over the business cycle. Panels (b) and (c) of Figure 1 characterizes firm-aggregate covariance and firm-aggregate covariance over market value for productivity, sales, and profit growth in the cross section of firms. That analysis differs from, but compliments, recent and earlier empirical work characterizing the cross section of firms over the business cycle. The early work is most closely associated with Gertler and Gilchrist (1994), who find that small firms respond more than large firms to monetary policy events because, as they argue, small firms face greater credit market frictions. Chari et al. (2007), revisiting the Gertler and Gilchrist (1994) work, find in a longer time series that small and large firms respond in the same way to fluctuations in aggregate economic activity. Both studies use QFR manufacturing data. Other recent findings are mixed: Gourio (2007) finds that small firms' profits are more procyclical. Crouzet and Mehrotra (2017) find that small firms are slightly more responsive to the business cycle, but only more so than the largest half-percent of firms, and not because of financial frictions. Moscarini and Postel-Vinay (2009) focus on employment and find in multiple datasets that large firms are more responsive to the business cycle.

I contribute an additional data point and a new perspective on this line of empirical work. My methodology differs, in that I use an aggregate variance decomposition to highlight the dual nature of firm-aggregate covariance: on the one hand, firm-aggregate covariance measures firm contributions to aggregate variance, on the other, it measures firm exposure to aggregate fluctuations. I find that productivity, sales and profit growth rates at high-productivity firms covary more with aggregate growth rates, but less per dollar of market value. My interpretation is that high-productivity firms contribute more to aggregate variance, but expose investors to less business cycle risk.

Corporate diversification. The key mechanism in my model is business-line diversification, where a business line consists of a single technology and the consumption good it produces, and where high-productivity firms operate more business lines using technologies that few other firms use.

Empirical work suggests that high-productivity firms are indeed better diversified, both in terms of product lines and production methods. Bernard et al. (2010) document the prevalence of multi-product firms in U.S. manufacturing, and in regressions they find a positive correlation between product adding and firm-level productivity. Broda and Weinstein (2010) analyze ACNielson bar code data and find that large firms sell a far greater number and variety of products at the upc, brand, module, and product group levels. Large firms are on average high-productivity firms, so this work also suggests greater product diversification at high-productivity firms. Empirical evidence on technology use is scarce, but Dunne (1991) uses data from the U.S. Census Survey of Manufacturing Technology to estimate adoption probabilities for seventeen advanced technologies, and finds that large manufacturers were more likely than small to adopt each of the seventeen technologies in the survey.

In theoretical work, firm diversification often entails technological change. For Ansoff (1957), firms diversify when they sell new products in new markets, and diversification typically requires "new skills, new techniques, and new facilities." Frankel (1955) provides an early description of the costs and considerations associated with introducing new production methods alongside old, and Gort (1962) argues that diversifying firms typically enter fast growing industries with high rates of technological change. My model captures much of this technological diversification theory by allowing product diversification only through technology adoption, and by introducing a fixed cost for each technology that firms operate. In the name of simplicity, the model does not differentiate goods by industry (beyond consumption and capital) so it misses some of the richness of Gort's theory. It also misses strategic aspects of technology adoption, as emphasized by Reinganum (1981) and Fudenberg et al. (1983), for example.

Gollop and Monahan (1991) survey a literature on index measures of firm diversification. The best measures are sensitive to a firm's product count, the distribution of its sales across products, and the similarity of the products themselves. Unfortunately, Compustat segment data only allow for simple business line counts, a coarse measure of diversification with little cross-sectional variation. For example, high-productivity firms report on average 2.6 segments each, compared to 1.5 for low-productivity firms.

Finally, a large finance literature focuses on corporate diversification and firm value. Martin and Sayrak (2003) survey the literature and describe the prevailing view in the 1990s as one of diversified firms trading at discounts. With the availability of more granular data, this view is changing. For example, Villalonga (2004) compares Compustat data with U.S. Census data that allows for finer measures of diversification and finds that diversified firms trade at a premium relative to focused firms when diversification is measured in the Census segment data but not when measured in the coarser Compustat segment data. Wang (2012) looks at stock returns rather than firm value, and finds that diversified firms in Compustat have lower returns on average. He argues that diversification affects a firm's systemic risk though its growth options. The predictions of my theoretical model are consistent with the recent empirical work by Villalonga and Wang.

3 Theoretical Framework

I now construct a simple two-sector production economy that rationalizes the motivating evidence presented in the introduction. In the model, firms produce capital and differentiated consumption goods for a representative household. Capital is produced by a representative firm, while consumption goods are produced by a continuum of monopolistically competitive firms. The latter firms each produce a number of different consumption goods using a number of different technologies. The technologies are non-rivalrous, so any number of firms can use the same technology, and the varieties are differentiated by technology and producer, so different firms using the same technology produce different goods. Firms choose their technology sets from a continuum of available technologies, each technology is a distinct source of randomness, and technology is the only source of randomness in the model. In particular, there is no purely-aggregate source of randomness, though economic aggregates do still fluctuate randomly. The remaining primitive assumptions and equations of the model are given below, and propositions in Section 4 explain the main mechanisms and highlight key results.

3.1 Consumption Goods Producers

Firms indexed by $\omega \in \Omega$ compete monopolistically for household demand for consumption goods. Each firm uses multiple technologies, and each technology produces a distinct differentiated consumption good. The total mass of active firms will later be normalized to one.

3.1.1 Production

Firms use technologies that combine labor and capital to produce consumption goods at constant returns to scale. Each technology has its own random productivity multiplier, denoted $z_t(v)$. Technological productivity is the first of two productivity types in the model, and is the only source of randomness. The second productivity type is firm-specific and non-random, and denoted $z(\omega)$. One interpretation is that the firm-specific productivity reflects management. Bloom et al. (2016) document a wide dispersion in management practices across firms, and find evidence that this dispersion in management practices explains some of the observed dispersion in productivity across firms. Firm-specific productivity follows a

Pareto distribution over firms, with shape parameter κ and scale parameter set to one.[4] I assume Cobb-Douglas production functions, and write firm ω's output $y_t(v,\omega)$ of the variety created by technology v at time t as

$$y_t(v,\omega) = z(\omega)z_t(v)\big[k_t(v,\omega)\big]^{\alpha}\big[l_t(v,\omega)\big]^{1-\alpha}, \tag{3}$$

where parameter α controls the cost share attributed to the capital $k_t(v,\omega)$ and the labor $l_t(v,\omega)$ the firm uses to produce the good.

Finally, note that capital is homogeneous in this set-up, in contrast to traditional vintage capital models. Firms can move capital freely across technologies and combine it with labor in varying proportions. This assumption is analytically convenient, and increasingly plausible economically, in light of the increasing flexibility of manufacturing systems observed by Milgrom and Roberts (1990).[5]

3.1.2 Profit Maximization

Profits and prices are expressed in units of an aggregate consumption basket that will be specified in section 3.2. Firms take the wage w_t and the capital rental rate r_t as given, but act as monopolists in each of their differentiated goods, setting prices to maximize profits. Denote by $p_t(v,\omega)$ the price that firm ω sets for the variety it produces with technology v,

[4] Firms don't draw their productivities randomly from the Pareto distribution, as this can lead to measurability problems (**Uhlig:1996aa**; see Doob, 1953; Feldman and Gilles, 1985; Judd, 1985; Khan and Sun, 1999, for details and proposed solutions). Here, I assume firm-specific productivities were assigned deterministically at some point in the past. Thus, the Pareto distribution here lacks a probability interpretation and needn't integrate to one.

[5] In a future version I intend to extend the model by indexing capital according to vintage and restricting the use of capital to technologies of corresponding vintage. The extension would differ from the putty-clay assumption made in many vintage capital models, in that it would place no restriction on the proportions in which inputs are combined in production. It would, however, bring the model closer to vintage capital models in the style of Solow (1960), which tend to allow for easier aggregation. See Johansen (1959) for an early putty-clay model, or Boucekkine et al. (2011) for a recent survey of vintage capital models of both varieties.

and write the firm's gross profit from producing $y_t(v,\omega)$ units of the variety as

$$\pi_t(v,\omega) = p_t(v,\omega)y_t(v,\omega) - r_t k_t(v,\omega) - w_t l_t(v,\omega). \tag{4}$$

A firm's total gross profit $\Pi_t(\omega)$ equals the sum of gross profits from each of its varieties: $\Pi_t(\omega) = \int_{\mathcal{V}(\omega)} \pi_t(v,\omega)\lambda(dv)$. Firms are owned by the representative household, so they use the household's stochastic discount factor $m_{t,s}$ to discount expected future profits. They maximize value by choosing optimal factor inputs and prices for each differentiated good, subject to the production function given by equation (3), and subject to downward-sloping household demand given later by equation (39). Let $\mathcal{V}(\omega)$ represent the set of technologies that firm ω uses, and write the firm's decision problem as

$$\begin{aligned} \max_{\left\{\begin{smallmatrix} k_t(v,\omega), \\ l_t(v,\omega), \\ p_t(v,\omega) \end{smallmatrix}\right\}_{v\in\mathcal{V}(\omega)}} \quad & \Pi_t(\omega) = \int_{\mathcal{V}(\omega)} \pi_t(v,\omega)\lambda(dv) \\ \text{s.t.} \quad & (3) \text{ and } (39) \quad \forall\, v \in \mathcal{V}(\omega). \end{aligned} \tag{5}$$

3.1.3 Technology Choice

Firms choose their technology sets, denoted $\mathcal{V}(\omega)$, from a fixed set $\mathcal{V} = [\,\underline{v}, \infty)$ of available technologies, and make their chooses one period in advance. The parameter $\underline{v}$ is exogenous and positive. There is no cost to adopting or abandoning a technology, but firms pay a fixed cost in each period that they operate a technology, paid in units of capital. The period fixed cost for technology v is given by:

$$f_{t+1}(v) = \frac{Y_{t+1}}{\mu} v^{\gamma}, \tag{6}$$

where γ governs the availability of technologies with low fixed operating costs, and where the specific functional form was chosen for analytical tractability. The coefficient Y_{t+1} simplifies the model dramatically: it causes period fixed costs to rise and fall with aggregate production Y_t, rendering the technology choice problem—and therefore all uncertainty in the model—completely static. Dynamic uncertainty is an attractive feature, but beyond the scope of this paper. In on-going work, I relax the simplifying assumption and study the dynamics of technology adoption, obsolescence, and endogenous uncertainty in an otherwise similar environment.

Under the present simplifying assumptions, the rule for choosing technology sets that maximize expected profit is immediate: Operate any technology v that satisfies

$$\mathrm{E}_t\Big[m_{t,t+1}\big(\pi_{t+1}(v,\omega) - f_{t+1}(v)\big)\Big] > 0. \tag{7}$$

Finally, an aim of this paper is to characterize the stochastic properties of firm-level and economy-wide productivity aggregates, but technical challenges arise when each firm's technology set is a continuous subset of $\mathcal{V}$. Aggregation then requires integrating over uncountable sets of random variables, and care must be taken to preserve randomness in the aggregates. To this end, I make the following assumption on technological productivity:

$$z_t(v)^{\theta-1} := \epsilon_{t,\lceil v \rceil} \quad \forall\ v \in \mathcal{V}, \tag{8}$$

with $\{\epsilon_{t,1}, \epsilon_{t,2}, \dots\}$ a *countable* set of random variables. Think of the $\epsilon_{t,n}$'s as fundamental technologies upon which production technologies represented by the $z_t(v)$'s are built. I make the following assumptions on the fundamental technologies: for all $n, m \in \mathbb{N}$ and $s, t \in \mathbb{Z}$, with $n \neq m$ and $s \neq t$, assume that $\mathrm{E}\big[\epsilon_{t,n}\big] = \mu_\epsilon$, $\mathrm{Var}\big(\epsilon_{t,n}\big) = \sigma_\epsilon^2$, and $\mathrm{Cov}\big(\epsilon_{t,n}, \epsilon_{t,m}\big) = \mathrm{Cov}\big(\epsilon_{s,n}, \epsilon_{t,n}\big) = 0$. This construction is a special case of the general construction developed in Al-Najjar (1995), and is specifically designed to preserve risk in continuum economies.[6]

3.2 Household

Each period, the representative household spends C_t on consumption, invests I_t in physical capital, and ownsa portfolio of firms, each valued at $V_t(\omega)$. To pay for its consumption and investments, the household sells to firms a fixed quantity of labor L at wage w_t; it rents to firms the physical capital K_t it owns at interest rate r_t; and it collects firms' net profits, where net profit is gross profit $\Pi_t(\omega)$ minus fixed costs $F_t(\omega)$. The budget constraint summarizes the household's sources and uses of funds. In units of consumption, write the constraint with sources on the left, and uses on the right:

$$\begin{aligned} w_t L + r_t K_t + \int_\Omega \big[\Pi_t(\omega) - F_t(\omega)\big] S_t(\omega)\lambda(d\omega) \\ = C_t + I_t + \int_\Omega V_t(\omega)\big[S_{t+1}(\omega) - S_t(\omega)\big]\lambda(d\omega), \end{aligned} \tag{9}$$

where $S_t(\omega)$ represents the household's firm ownership share. Capital depreciates at rate δ, and therefore evolves according to

$$K_{t+1} = I_t + (1-\delta)K_t. \tag{10}$$

The household is impatient, risk averse, loves variety in consumption, and views all consumption goods as equally substitutable. I capture these preferences formally with a

[6] Al-Najjar considers collections of random variables f indexed by the measure space $(T, \mathcal{T}, \tau)$, where $T = [0,1]$ is a continuous parameter space, and $f_t = g_t + h_t$, with aggregate component $g_t = \sum_{k=1}^{\infty} \beta_k \eta_k$, $\{\eta_1, \eta_2, \dots\}$ a set of orthonormal random variables, and idiosyncratic component h_t such that $\mathrm{E}[xh_t] = 0$ τ-a.e. for any random x defined on the same probability space as h_t. In Al-Najjar's notation, I consider the case of $h_t = 0$ τ-a.e., and $\beta_k := \beta_k(t) = 1$ if $k-1 < t \leq k$ and zero otherwise. Here, $\mathcal{V}$ corresponds to T, v to t, $z_t(v)$ to g_t, and $\epsilon_{s,n}$ to η_k. It is worth noting that the special case I consider extends trivially using Al-Najjar's construction to cases that feature idiosyncratic shocks to individual technologies or individual firms, and to cases where shocks are correlated across individual technologies.

logarithmic period utility function defined over a Dixit and Stiglitz (1977) aggregate of differentiated goods and discounted at rate β over time.[7] The set-up allows the household to allocate resources in two stages. In the first stage, the household allocates resources between consumption, physical capital, and firm ownership to maximize utility:

$$
\begin{aligned}
\max_{\left\{\substack{C_s,\\ K_{s+1},\\ S_{s+1}(\omega)}\right\}_{\substack{s\in\mathcal{T}_t,\\ \omega\in\Omega}}} \quad & U_t = \mathrm{E}\left[\sum_{s=t}^{\infty}\beta^{s-t}\ln(C_s)\right] \\
\text{s.t.} \quad & (9) \text{ and } (10) \quad \forall\ s \geq t.
\end{aligned}
\tag{11}
$$

In the second stage, the household optimally allocates resources among differentiated goods per unit of consumption expenditure:

$$
\begin{aligned}
\max_{\left\{c_t(v,\omega)\right\}_{\substack{v\in\mathcal{V},\\ \omega\in\Omega}}} \quad & C_t = \left[\int\limits_{\Omega}\int\limits_{\mathcal{V}(\omega)} \left[c_t(v,\omega)\right]^{\frac{\theta-1}{\theta}}\lambda(dvd\omega)\right]^{\frac{\theta}{\theta-1}} \\
\text{s.t.} \quad & 1 = \int\limits_{\Omega}\int\limits_{\mathcal{V}(\omega)} p_t(v,\omega)c_t(v,\omega)\lambda(dvd\omega)
\end{aligned}
\tag{12}
$$

I discussion the household problem and provide further derivations in appendix A.1.1.

3.3 Capital Producer

For convenience, I separate capital production from consumption goods production in distinct sectors. Doing so reduces the number of state variables in the technology choice problem, and it simplifies aggregation. Separating production serves no purpose beyond this convenience, so a basic specification of capital production will suffice.

[7] Recall that logarithmic utility captures a special case of risk aversion and intertemporal substitutability, where both the Arrow-Pratt measure of relative risk aversion and the elasticity of intertemporal substitution equal one.

A representative and privately-owned firm supplies the household with capital. As in the consumption goods sector, the firm here combines labor and capital in a constant-returns-to-scale production function with a stochastic productivity multiplier. The productivity multiplier is an aggregate of the technological and firm-specific productivities of consumption goods producers. Because the consumption and capital goods producers are subject to the same aggregate fluctuations, the price of the consumption basket correlates perfectly with the price of capital. The relative price therefore equals the constant markup of consumption goods producers; I normalize this price to one by choice of capital units.

The capital producer chooses capital and labor inputs to maximize profit, taking prices as given. Write the production function as $\tilde{I}_t = Z_t(k_t)^{\alpha}(l_t)^{1-\alpha}$, the gross profit function as $\pi_t = \tilde{I}_t - r_t k_t - w_t l_t$, and the profit maximization problem as:

$$\max_{k_t, l_t} \quad \pi_t, \tag{13}$$

where the maximization problem is subject to the production and gross profit functions above, and where productivity Z_t is specified in more detail in Proposition 4.2. See Appendix A.1.1 for optimality conditions from the decision problems in Equations (5), (11) and (13).

3.4 Equilibrium

An equilibrium is a set of consumption goods prices $\{p_t(v, \omega)\}_{\omega \in \Omega, v \in \mathcal{V}}$, factor market prices w_t and r_t, and firm values $\{V_t(\omega)\}_{\omega \in \Omega}$ at which the household budget constraint is satisfied, the consumption and capital goods markets clear, capital and labor factor markets clear, the stock market clears, optimality conditions in equations (32) to (34), (41) and (42) (found in appendix A.1.1) are satisfied, and firm technology sets $\{\mathcal{V}(\omega)\}_{\omega \in \Omega}$ contain all technologies that satisfy the adoption rule in equation (7) and none that violate it. See Appendix A.1.3 for details and steady-state equilibrium expressions.

4 Main Propositions

The propositions in this section communicate two nice features of the theoretical framework: first, that the framework is analytically tractable; and second, that the framework produces uncertainty and covariance endogenously. The proofs are straightforward but tedious, and I provide them with some discussion in Appendix A.1.2.

Proposition 4.1 shows that a minor modification to the model in Section 3 completely changes the interpretation of uncertainty as arising from fluctuations in demand rather than from fluctuations in technological productivity. The proposition highlights the flexibility of the model, showing that random fluctuations in demand produce results that are similar in many ways to those in the model with random fluctuations in technological productivity. Proposition 4.2 characterizes aggregation, highlighting the analytical tractability of the model. It states that the model can be aggregated in two ways: over technologies, and over firms; and that special productivity averages completely summarize the economy's technological and firm-specific heterogeneity. Proposition 4.3 characterizes firm-level technology sets. It states that profit-maximizing firms use all available technologies below a firm-specific cost threshold, and that high-productivity firms have higher cost thresholds. Conveniently, each firm's cost threshold is enough to fully characterize its technology set. Proposition 4.4 highlights how firm technology choices endogenize uncertainty in the model. It gives closed-form expressions for the endogenous first and second moments of firm and aggregate productivity distributions, assuming firms operate the technologies they would choose in non-stochastic steady state. As it happens, these technology sets are first-order approximations to the sets firms would choose in a stochastic world. Proposition 4.5 characterizes comovement, and highlights qualitative features the model shares with the data. It gives an exact closed-form expression for endogenous firm-aggregate productivity covariance, and an approximate expression for firm-aggregate productivity covariance over market value. Finally, proposition 4.7 shows how covariance risk affects stock returns. It gives an approximate expression for expected excess returns in terms of firm-aggregate productivity covariance over market value, and states that expected returns are lower for high-productivity firms.

4.1 Random Fluctuations in Demand

Random fluctuations in demand may influence firm-level productivity estimates through prices, because the productivity estimates are based on firms' reported revenues—the product of prices and quantities (see De Loecker et al., 2017, for a recent discussion). Demand-induced fluctuations in firm-level revenue can be difficult to distinguish from supply-induced fluctuations, and I make no attempt here. Instead, proposition 4.1 states that a modified model—with shocks to preferences rather than technologies—produces the same firm-level covariance structure as the model with technology shocks presented in section 3. The point is that covariance arises because firms choose their business risks, not because those risks come specifically from supply or demand.

Proposition 4.1 (Random Fluctuations in Demand)**.** *Let $z_t(v)$ now be a random preference multiplier. Replace the stochastic production function in equation (3) with equation (3′) below, and the non-stochastic preferences in equation (12) with equation (12′) below:*

$$y_t(v,\omega) = z(\omega)\big[k_t(v,\omega)\big]^{\alpha}\big[l_t(v,\omega)\big]^{1-\alpha}, \tag{3′}$$

$$C_t = \left[\int_{\Omega}\int_{\mathcal{V}(\omega)} \big[z_t(v)c_t(v,\omega)\big]^{\frac{\theta-1}{\theta}} \lambda(dv d\omega)\right]^{\frac{\theta}{\theta-1}}. \tag{12′}$$

Then the propositions of this section remain true after derivation of the appropriate stochastic household demand curve for individual varieties. The proof is in the appendix.

One interpretation of the modified model is that fluctuations in preferences are specific to product features, and product features are specific technologies. I return to the original equations (3) and (12) and the supply-side interpretation for the remainder of the exposition.

4.2 Aggregation

To characterize the endogenous probability distributions of firm-level and economy-wide productivity, aggregation is necessary. Fortunately, the model aggregates easily, both over sets of technologies and over sets of firms, despite the heterogeneity in each of these sets. It is therefore possible to find nice analytical expressions for many endogenous variables at different levels of aggregation. For instance, production can be viewed at the differentiated-good level, the firm level, or the economy-wide level, and can be expressed in each case as a Cobb-Douglas production function of appropriately-aggregated capital and labor. The model is thus quite useful for studying microeconomic sources of aggregate fluctuation. Proposition 4.2 characterizes aggregates in terms of special productivity variables. The aggregation strategy I employ was first developed by Houthakker (1955), further developed by Melitz (2003) to study trade, and further still by Ghironi and Melitz (2005) to study macroeconomic dynamics and trade. Here, the aggregation occurs in two stages.

Proposition 4.2 (Aggregation)**.** *A productivity aggregate over technologies summarizes all of the technological heterogeneity within an individual firm ω:*

$$Z_t(\omega) = \left[\int_{\mathcal{V}(\omega)} \left[z(\omega) z_t(v) \right]^{\theta-1} \lambda(dv) \right]^{\frac{1}{\theta-1}}. \tag{14}$$

A productivity aggregate over firms summarizes all of the firm-specific and technological *heterogeneity within the consumption goods sector:*

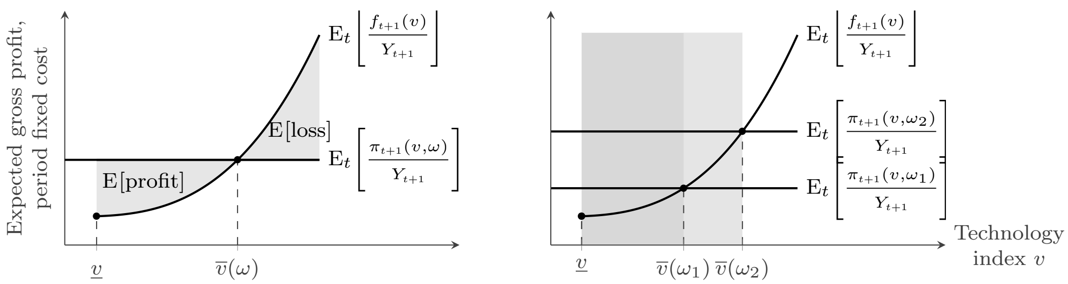
$$Z_t = \left[\int_{\Omega} Z_t(\omega)^{\theta-1} \lambda(d\omega) \right]^{\frac{1}{\theta-1}}. \tag{15}$$

Aggregate factor demands, production, and profit can be written in terms of aggregate productivities and variables that either do not vary across firms, in the case of firm aggregates, or do not vary across firms or technologies, in the case of economy-wide aggregates. The proof is in the appendix.

The aggregate expressions for factor demands, output, and profit take simple forms, and suggest a close relationship between the economy here with multi-product, multi-technology firms, and simpler production economies with single-technology, single-product firms, or with a single representative firm. For instance, the basic Cobb-Douglas production structure is preserved in aggregation. The heterogeneous technologies do, however, constrain the stochastic processes driving productivity aggregates; the constraints force the model to capture qualitative features of cross-sectional evidence from Compustat on firm-level covariance, and first and second moments of firm-level productivity. Again, the model captures these cross-sectional features endogenously by presenting firms with a technology choice problem.

4.3 Technology Choice

Firms choose their technology sets in the model, and because technology is the only source of randomness, the choices firms make ultimately determine the probability distributions of random productivity shocks to firms and the aggregate economy. The technology choice problem is static under the simplifying assumptions made on the primitives of the model. Firms operate any technology that they expect will earn them positive net profit in the



(a) Technology choice problem **(b) Overlapping technology sets**

Figure 2 – **An illustration of the technology choice problem facing firms.** Figure 2(a) on the left illustrates the technology choice problem for firm ω. The vertical axis measures costs and benefits associated with firm ω's operation of different technologies. The horizontal axis represents the set of available technologies, arranged left to right from least to most expensive in terms of fixed costs. The horizontal expected discounted gross profit curve represents firm ω's expected benefit from operating the available technologies. The upward-sloping fixed cost curve represents the fixed cost associated with each technology. Firm ω's technology set is determined by the intersection of the expected gross profit curve and the fixed cost curve at point $\overline{v}(\omega)$. The firm can profitably produce only with technologies to the left of this point. Figure 2(b) on the right illustrates the technology choice problems of two firms: firm ω_1 with low productivity and ω_2 with high productivity. Note that the technology sets of these two firms overlap. Note in particular that the low-productivity firm's technology set is a proper subset of the high-productivity firm's technology set, so that, if only these two firms exist, there are some technologies that only the high productivity firm ω_2 operates, indicated by the lighter shaded region between the threshold points $\overline{v}(\omega)$ and $\overline{v}(\omega_1)$.

following period, after paying fixed operating costs in units of physical capital. The fixed cost differs across technologies, low for some, high for others, and most firms operate only a subset of the technologies available in $\mathcal{V}$. Because some firms have higher firm-specific productivity, they are able to operate profitably at higher fixed costs, and therefore choose to operate a greater number of technologies. Proposition 4.3 characterizes the technology choices that individual firms make.

Proposition 4.3 (Technology sets)**.** *In non-stochastic steady state, any firm ω with productivity $z(\omega) \geq \underline{z}$ chooses technology set $\mathcal{V}(\omega) = \left\{v \in \mathcal{V} : \underline{v} \leq v \leq \overline{v}(\omega)\right\}$, where the endogenous* *cut-offs $\underline{z}$ and $\overline{v}(\omega)$ are given by:*

$$\underline{z} = \left(\frac{\theta}{\mu_\epsilon}\right)^{\frac{1}{\theta-1}} \tag{16}$$

$$\overline{v}(\omega) = \left(\frac{\mu_\epsilon}{\theta}\right)^{\frac{1}{\gamma}} z(\omega)^{\frac{\theta-1}{\gamma}}. \tag{17}$$

Firms with $z(\omega) < \underline{z}$ do not produce. Under parameter restrictions, firms ω_1 and ω_2 with productivities $\underline{z} < z(\omega_1) < z(\omega_2)$ choose technology sets such that $\mathcal{V}_t(\omega_1) \subset \mathcal{V}_t(\omega_2)$. The above cut-offs are also first-order approximate to those that obtain in a stochastic environment. The proof is in the appendix.

Panel (a) of Figure 2 illustrates the technology choice problem of an arbitrary firm ω. Available technologies are arranged along the horizontal axis, and expected gross profit and operating cost measured on the vertical axis. The assumptions made on the fundamental productivity shocks $\epsilon_{s,\lceil v \rceil}$ in Section 3 imply that the expected gross profit curve is horizontal to a first-order approximation, while the operating cost curve is assumed to increase in the technology index v. The intersection of gross profit and operating cost curves marks firm ω's cost threshold, and the firm cannot profitably diversify into new technologies and business lines above this cost threshold.

4.4 Firm and Aggregate Productivity

The expected values and variances of the random firm-level and economy-wide productivity aggregates are endogenous, because they depend on the technology sets that firms choose to operate, and firms make this choice endogenously. I use the cost threshold from the technology choice problem in Proposition 4.3 to derive explicit expressions for the expected values and variances of firm-level and economy-wide productivity in Proposition 4.3.

Proposition 4.4 (Productivity First and Second Moments)**.** *Let technology sets be those that firms choose in non-stochastic steady state. Then the first and second moments of firm-level productivity are given by* $\mu(\omega)$ *and* $\sigma^2(\omega)$*, respectively:*

$$\mu(\omega) = \mu_\epsilon z(\omega)^{\zeta_{\mu\omega 1}} \left[\left(\frac{z(\omega)}{\underline{z}} \right)^{\zeta_{\mu\omega 2}} - 1 \right], \tag{18}$$

$$\sigma^2(\omega) = \sigma_\epsilon^2 z(\omega)^{\zeta_{\sigma\omega 1}} \left[\left(\frac{z(\omega)}{\underline{z}} \right)^{\zeta_{\sigma\omega 2}} - 1 \right]. \tag{19}$$

The first and second moments of aggregate productivity are given by μ *and* σ^2*, respectively:*

$$\mu = \mu_\epsilon \zeta_{\mu 1} \underline{z}^{\zeta_{\mu 2}}, \tag{20}$$

$$\sigma^2 = \sigma_\epsilon^2 \zeta_{\sigma 1} \underline{z}^{\zeta_{\sigma 2}}. \tag{21}$$

Under parameter restrictions, the first and second moments of all productivity aggregates are positive and finite, and for any firms ω_1 *and* ω_2 *with* $z(\omega_1) < z(\omega_2)$*, it holds that* $\mu_t(\omega_1) < \mu_t(\omega_2)$ *and* $\sigma_t^2(\omega_1) < \sigma_t^2(\omega_2)$*. The proof is in the appendix.*

4.5 Productivity Comovement

Covariance in the model arises from overlapping technology sets: when firms use similar technologies, they are subject to similar fluctuations in technological productivity, and their productivities covary. Panel (b) of Figure 2 illustrates this effect. Because high-productivity firms have larger technology sets and produce at greater scale, they have more overlap with other firms, and higher covariances.

High-productivity firms are also more profitable than other firms, and so have higher market values, but above a low threshold, the ratio of covariance to market value falls in firm productivity. The ratio is falling because more productive firms use some technologies that few other firms use; these technologies generate profit for the firm, which raises market value, but contributes little to the covariance between firm and aggregate productivity. As a firm's productivity approaches the productivity cut-off $\underline{z}$ from above, both its covariance and its market value race to zero.

Proposition 4.5 (Firm-to-Aggregate Productivity Covariance)**.** *Let technology sets be those that firms choose in the non-stochastic steady state. Then the covariance between firm and aggregate productivity, denoted by* $\sigma_{\omega\Omega}(\omega) = \mathrm{Cov}\left(Z_t(\omega)^{\theta-1}, Z_t^{\theta-1}\right)$, *is given by*

$$\sigma_{\omega\Omega}(\omega) = z(\omega)^{\theta-1}\zeta_{\omega\Omega 1}\left[1 - \left(\frac{\underline{z}}{z(\omega)}\right)^{\zeta_{\omega\Omega 2}}\right] \tag{22}$$

The covariance between firm and aggregate productivity, expressed as a fraction of firm market value, is approximated to a first order by

$$\frac{\sigma_{\omega\Omega}(\omega)}{V_t(\omega)} \approx \frac{1}{Y_t}\left(\frac{\zeta_{\omega\Omega 1}\left[1 - \left(\frac{\underline{z}}{z(\omega)}\right)^{\zeta_{\omega\Omega 2}}\right]}{\zeta_{V1}\left(\frac{z(\omega)}{\underline{z}}\right)^{\zeta_{V2}} + \zeta_{V3}\left(\frac{1}{z(\omega)}\right)^{\zeta_{V4}} - \left(\frac{1}{\underline{z}}\right)^{\zeta_{V4}}}\right). \tag{23}$$

Under parameter restrictions, covariance-over-value falls for all $z(\omega)$ above a threshold. The ratio also falls in the level of aggregate output. The proof is in the appendix.

In the model, firms with similar managerial productivities choose similar technology sets, so these firms have more highly-correlated productivities. As panel (d) of Figure 2 illustrates, this effect can be seen in the data.

Proposition 4.6 (Firm-to-Firm Productivity Covariance)**.** *Let technology sets be those that firms choose in the non-stochastic steady state, and let $Z_t(\omega_1)$ and $Z_t(\omega_2)$ be firm productivities for firms ω_1 and ω_2, where $Z_t(\omega_1) < Z_t(\omega_2)$. Then the correlation between* *firm productivities is given by*

$$\mathrm{Corr}\big(Z_t(\omega_1), Z_t(\omega_2)\big) = blah, \tag{24}$$

and the correlation $\mathrm{Corr}\big(Z_t(\omega_1), Z_t(\omega_2)\big)$ *is decreasing in the distance between productivities,* $|z(\omega_1) - z(\omega_2)|$. *The proof is in the appendix.*

4.6 Stock returns

Firm-aggregate covariance over market value measures firm-level systemic risk in the model. In theory, a risk-averse investor should be willing to accept lower returns from high-productivity firms, because the activities of these firms covary less with aggregate activity, relative to the discounted future profit investors expect. As in the classical capital asset pricing models, and the consumption based models, I am able to directly express expected stock returns in terms of covariance—in this case, firm-aggregate productivity covariance over market value.

Proposition 4.7 (Stock returns)**.** *Let technology sets be those that firms choose in the non-stochastic steady state. Then firm* ω*'s expected excess return is approximated to a second order by*

$$\mathrm{E}_t\big[r_{t+1}(\omega) - r_{f,t+1}\big] \approx \zeta_{r1}\frac{\mu(\omega)}{V_t(\omega)} + \zeta_{r2}\frac{\sigma_{\omega\Omega}(\omega)}{V_t(\omega)}, \tag{25}$$

where I define firm ω*'s return as* $r_t(\omega) = \big[V_{t+1}(\omega) + \Pi_{t+1}(\omega) - F_{t+1}(\omega)\big]/V_t(\omega)$*, and the risk-free rate as* $r_{f,t} = m^{-1}_{t,t+1}$*. Under parameter restrictions, expected excess returns decrease in firm productivity* $z(\omega)$ *for all* $z(\omega)$ *above a threshold. The proof is in the appendix.*

The propositions in this section explain the model's mechanism, and highlight key results. In particular, the propositions illustrate how the technology choice mechanism leads to endogenous first and second moments of firm-level and aggregate productivity, and endogenous covariance between firm and aggregate productivity. The model is able to capture many of the features of firm-level covariance documented in the motivating evidence in Section 1, and the propositions also show how covariance affects systemic risk and stock returns.

5 Empirical Framework

This section describes the empirical framework I use to develop the motivating evidence in section 1, and the regression framework I use to check key predictions of the model. Section 5.1 describes the Compustat data and other data sources. Section 5.2 describes the productivity estimation procedure, which I follow from Olley and Pakes (1996) and İmrohoroğlu and Tuzel (2014). Section 5.3 describes the rolling-window covariances and other calculations used to produce figure 1. Table 2 reports the statistics that underlie figure 1. Section 5.4 describes the regression framework. The regressions control for firm-level differences in financial strength, fixed firm-specific characteristics, and common aggregate shocks.

5.1 Data Description

For accounting data I use the WRDS Compustat North America Fundamentals Annual database, which, after cleaning, covers 67,693 observations on 7,462 distinct firms over the period 1966–2015. Figure 3 illustrates some features of the sample. The Compustat data cover foreign and domestic firms that are or were public in the United States.

The accounting variables used in productivity estimation and aggregate variance decompositions are employment (`EMP`); net property, plant and equipment (`PPENT`) as a measure of physical capital; depreciation (`DP`) and accumulated depreciation (`DPACT`) to estimate the age of the capital stock; net sales (`SALE`), which I refer to as sales; operating income before before depreciation (`OIBDP`), which I refer to as profit; and fiscal year closing share price (`PRCC_F`) and common shares outstanding (`CSHO`) to compute market value. I use the following additional variables to construct controls for financial strength in regressions: total debt (`DT`) and common equity (`CEQ`) to compute leverage; interest expense (`INT`) to compute the interest coverage ratio; and cash and short term investments (`CHE`), receivables (`RECT`), and current liabilities (`LCT`) to compute the quick ratio. I follow İmrohoroğlu and Tuzel (2014) and Covas and den Haan (2011) in cleaning the Compustat data: I drop financial and utilities firms, observations prior to 1961, observations with missing values on any of the variables used in productivity estimation or rolling-window covariances, firms involved in large mergers, and the smallest 10% of firms by market value. I use data on nominal GDP, and GDP and non-residential investment deflators from the Bureau of Economic Analysis, as well as Social Security Administration data on national average wage.

5.2 Productivity Estimation

For consistency with an established literature, and for consistency with my theoretical model, I characterize the firm cross-section using estimated total factor productivity, rather than, say, sales, employment, or market value.[8] I follow the procedures in Olley and Pakes (1996) and İmrohoroğlu and Tuzel (2014), and estimate a Cobb-Douglas production function in log form:[9]

$$\ln(Y_{\omega,t}) = \alpha_0 + \alpha_K \ln(K_{\omega,t}) + \alpha_L \ln(L_{\omega,t}) + \ln(Z_{\omega,t}), \tag{26}$$

[8] An empirical literature documents substantial differences in total factor productivity across firms: Baily et al. (1992), Bartelsman and Doms (2000), and Foster et al. (2001) provide evidence for U.S. manufacturing, Olley and Pakes (1996) provide evidence for the telecommunications industry and develop a now widely-used productivity estimation procedure. Bartelsman, Haltiwanger, et al. (2009) provide cross-country evidence. Heterogeneous productivity also plays an important role in theory: firm-level productivity shocks drive a class of models used to study industry dynamics, beginning with Jovanovic (1982) and Hopenhayn (1992)

[9] I thank Selale Tüzel for making her productivity estimation code available online, key parts of which I have used in this project.

Table 1 – Summary Statistics for Compustat Firms. Firms are grouped into productivity deciles, each decile forming a column. All statistics are averaged or aggregated within decile, then averaged over time. Averages are reported over the forty-year period 1976–2015, and the two twenty-year periods 1976–1995 and 1996–2015; Compustat segment data is unavailable prior to 1976. The first row of each panel shows decile productivity averages relative to the full-sample average; averages are taken over firms, the decile average is expressed relative to the full-sample average, then averages are taken over time. The next three row show variance, firm-aggregate covariance, and firm-aggregate covariance-over-value, again expressed as yearly decile averages relative to yearly full-sample average, and averaged over time. Row five shows the average number of segments reported by firms in each decile. The last two rows show aggregate decile shares of total employment, total sales, and total profit, again averaged over time. Each of the last two rows sums to one hundred plus rounding errors.

	1976–2015	Low	2	3	4	5	6	7	8	9	High
Productivity		0.39	0.50	0.58	0.66	0.74	0.84	0.96	1.13	1.42	2.78
— variance		0.01	0.01	0.01	0.01	0.02	0.16	0.06	0.51	1.21	8.00
— covariance		0.01	0.06	0.06	0.11	0.14	0.25	0.48	0.84	1.67	6.37
— cov-over-val		2.46	2.54	1.09	1.26	0.75	0.65	0.53	0.30	0.08	0.34
Number of segments		1.53	1.53	1.62	1.69	1.75	1.88	2.04	2.21	2.49	2.55
Employment share		0.39	0.83	1.42	2.19	3.50	5.42	8.82	14.65	25.44	37.34
Sales share		0.26	0.55	0.92	1.46	2.23	3.51	6.21	11.27	21.29	52.29
Profit share		-0.02	0.10	0.27	0.56	1.05	1.90	3.58	7.47	18.33	66.78
	1996–2015										
Productivity		0.33	0.45	0.54	0.62	0.70	0.80	0.92	1.10	1.42	3.12
— variance		0.01	0.01	0.01	0.00	0.04	0.30	0.06	0.48	2.21	6.89
— covariance		-0.01	0.06	0.06	0.10	0.13	0.23	0.51	0.87	2.11	5.95
— cov-over-val		2.65	2.76	0.79	1.04	0.79	0.75	0.60	0.22	0.03	0.38
Number of segments		1.39	1.40	1.49	1.56	1.61	1.70	1.75	1.94	2.19	2.18
Employment share		0.30	0.77	1.46	2.39	3.96	5.97	10.04	16.16	27.46	31.48
Sales share		0.19	0.50	0.88	1.44	2.32	3.58	6.74	12.17	23.89	48.29
Profit share		-0.07	0.03	0.15	0.41	0.98	1.67	3.44	7.37	18.83	67.18
	1976–1995										
Productivity		0.45	0.55	0.62	0.70	0.78	0.88	1.00	1.16	1.41	2.44
— variance		0.01	0.00	0.01	0.02	0.01	0.03	0.06	0.54	0.21	9.12
— covariance		0.03	0.06	0.07	0.13	0.14	0.27	0.46	0.82	1.24	6.79
— cov-over-val		2.27	2.32	1.40	1.48	0.71	0.55	0.47	0.39	0.12	0.30
Number of segments		1.67	1.66	1.76	1.83	1.89	2.07	2.32	2.47	2.79	2.91
Employment share		0.47	0.89	1.38	1.99	3.04	4.87	7.60	13.14	23.42	43.19
Sales share		0.33	0.61	0.96	1.48	2.14	3.44	5.67	10.38	18.69	56.29
Profit share		0.02	0.17	0.38	0.71	1.12	2.12	3.73	7.56	17.82	66.38

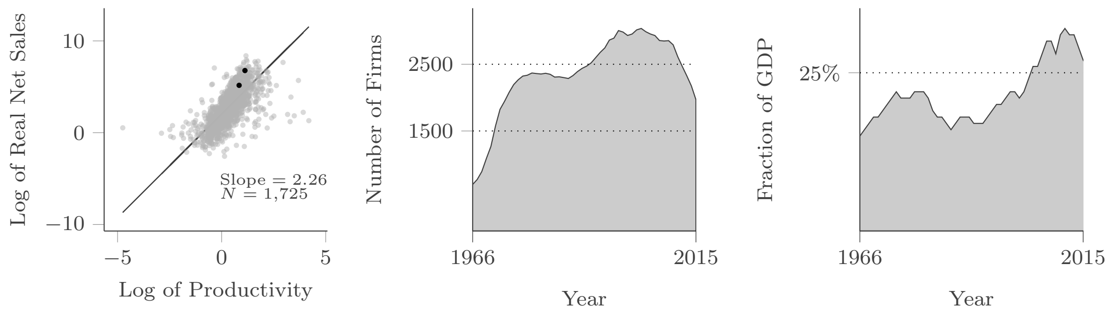


Figure 3 – Descriptive Statistics for the Compustat Sample, 1966–2015. The scatter plot on the left relates log firm size to log total factor productivity for the year 2015. Size is measured by net sales, in millions of 2009 dollars, and total factor productivity is estimated by the Olley and Pakes (1996) method. The black dots, from left to right, are Starbucks and Boeing. I apply the logarithmic transformation because both size and productivity distributions are highly skewed to the right. The middle figure plots the number of firms in the Compustat sample per year, and shows this number increasing rapidly in the early part of the sample. The rapid rise is partly due to the addition of NASDAQ in 1973, as Fama and French (1992) report. The figure on the right plots aggregate real value added for the Compustat sample as a fraction of U.S. real GDP. This fraction tends to rise with the number of firms in the sample. I drop financial and utilities firms, observations prior to 1961, observations with missing values on any of the variables used in productivity estimation or rolling-window covariances. I also drop firms in large mergers, and the smallest 10% of firms by market value.

where residual $Z_{\omega,t}$ is firm-level total factor productivity. The estimation procedure assumes that firms can partially forecast their future productivity, and controls for the simultaneity bias that arises from the forecasts; the procedure also controls for the selection bias that arises because of firm entry and exit in the Compustat sample. Finally, the procedure includes controls for time-industry effects. These measures are intended to reduce the biases and industry-level effects that would otherwise influence the production function parameter estimates. Appendix A.2.1 describes the estimation procedure in detail.

5.3 Aggregate Variance Decomposition

The motivating evidence for this paper derives from the aggregate variance decomposition in equation (1). In this section, I describe the decomposition in more detail, and apply the decomposition to Compustat data, using the total factor productivity estimates from the previous section to characterize the cross-section of firms. Table 2 summarizes the results of the decomposition for firms grouped by decade and productivity decile, for productivity, sales, and profit growth.

I compute sample variances and covariances in rolling windows for all firm pairs in my Compustat sample, and all available years. For firm variable x_ω, rolling window $\mathcal{W}_t = \{t-w,\ \ldots,\ t-1,\ t\}$, and ω_1, ω_2 two firm indices, the sample variance and pairwise covariances are given by:

$$\mathrm{Var}_t(x_\omega) = \frac{1}{w} \sum_{s \in \mathcal{W}_t} (x_{\omega,s} - \overline{x}_{\omega,s})^2, \tag{27}$$

$$\mathrm{Cov}_t\big(x_{\omega_1}, x_{\omega_2}\big) = \frac{1}{w} \sum_{s \in \mathcal{W}_t} (x_{\omega_1,s} - \overline{x}_{\omega_1,s})(x_{\omega_2,s} - \overline{x}_{\omega_2,s}). \tag{28}$$

I choose a window length close to the average length of the post-war U.S. business cycle, measured peak to peak. The average length is 68.5 months using National Bureau of Economic Research dates, and I round up to six years because the Compustat data is annual.[10] The backward-looking window prevents future information from influencing the variance and covariance estimates, because future information would have been unavailable to investors trying to gauge at a point in time the risk in a firm's future profit stream.

To compute the rolling window covariances, I restrict the sample of firms each period to include only those firms with a sufficient history of non-missing observations. Denoting this set of firms $\Omega^n_{\mathcal{W}_t}$ (n for non-missing), the variance of aggregate variable $X = \sum_{\Omega^n_{\mathcal{W}_t}} x_\omega$

[10]The NBER dates can be found at http://www.nber.org/cycles.html as of August 2018. Longer windows give more stable sample covariances, as you would expect, but they also reduce the number of firms in the sample, because firms with too few within-window observations must be excluded. In practice, varying the window length between five and ten years makes little difference to the main conclusions because the panel width is large.

becomes

$$
\begin{aligned}
\mathrm{Var}_t(X) &= \mathrm{Var}_t\left(\sum_{\Omega^n_{\mathcal{W}_t}} x_\omega\right) \\
&= \sum_{\Omega^n_{\mathcal{W}_t}} Var_t(x_\omega) + \sum_{\Omega^n_{\mathcal{W}_t}} \sum_{\Omega^n_{\mathcal{W}_t}\setminus\{\omega\}} \mathrm{Cov}_t(x_\omega, x_{\omega'}).
\end{aligned} \tag{29}
$$

I use the subsample with non-missing values because firm entry, exit, and missing data are common in Compustat and problematic for the decomposition. While consistent with my model and with previous work (Comin and Mulani, 2004), requiring consecutive years of non-missing observations is a costly convenience: it omits an important source of aggregate variance, and it biases the sample of firms.[11] Entry and exit are beyond the scope of this paper, but I consider them in a companion paper.

The rolling-window approach has been employed in the economics literature by Comin and Mulani (2004) and Forbes (2012), and does have some advantages: First, it adds a time dimension to the covariances that would be absent if they were computed over the full sample period, and second, it limits the practical problem of characterizing differences in covariance across high-productivity and low-productivity firm groups, when firms frequently move between groups—a problem known as reclassification bias, and discussed in Moscarini and Postel-Vinay (2009).

Table 2 reports covariance statistics by decade, and for firms sorted into productivity deciles. Pooling by decile is common in the finance literature (see Fama and French, 1992; Fama and French, 2008; İmrohoroğlu and Tuzel, 2014; Fama and French, 2016, for other

[11] Entry and exit as a source of variance has recently been emphasized by Ghironi and Melitz (2005), Bilbiie et al. (2012), Carvalho and Grassi (2015), and Clementi and Palazzo (2016). To see the implications of omitting it here, consider aggregate variable $X' = \sum_{\Omega_{\mathcal{W}_t}} x_\omega$, where $\Omega_{\mathcal{W}_t}$ is the set of firms with at least one observation in window $\mathcal{W}_t$. Now, $\mathrm{Var}_t\big(X'\big) = \mathrm{Var}_t(X) + \mathrm{Var}_t\Big(\sum_{\Omega^m_{\mathcal{W}_t}} x_\omega\Big) + 2\,\mathrm{Cov}_t\Big(X, \sum_{\Omega^m_{\mathcal{W}_t}} x_\omega\Big)$, where $\Omega^m_{\mathcal{W}_t}$ is the set of firms with missing values, and where $\Omega^n_{\mathcal{W}_t}$ and $\Omega^m_{\mathcal{W}_t}$ constitute a partition of $\Omega_{\mathcal{W}_t}$. My procedure ignores the second two terms in the aggregate variance expression.

examples). Table 3a reports the average fraction of aggregate variance explained by firm-level pairwise covariances, averaged over decades. It shows that over 80% of variance in aggregate productivity, sales, and profit growth is explained by firm-level covariance in all decades since 1966 for all three variables. Table 3b reports firm-level contributions to aggregate variance for the cross-section of firms. Specifically, the table reports share-weighted covariances between firm and aggregate productivity, sales, and profit growth rates, reported with and without dividing by market value. The reported statistics are relative to the cross-sectional average each year, averaged within productivity decile, then over years. Table 3b shows that high-productivity firms contribute over six times what the average firm contributes to aggregate variance, but only around one-third as much per dollar of market value.

5.4 Regression Analysis

The regressions in this section test key predictions of the model. The mechanism in the model is business-line diversification, where a business line consists of a technology and the consumption good it produces. Covariance arises when firms have overlapping technology sets, and for high-productivity firms with many business lines, this overlap is larger. But because high-productivity firms use some technologies that few other firms use, the technologies add more to firm value than to firm-aggregate productivity covariance, so that high-productivity firms are less risky per dollar invested. The model makes two basic predictions based on this logic:

1. Ceteris paribus, covariance between firm and aggregate growth rates increases in firm-level total factor productivity.

2. Ceteris paribus, covariance between firm and aggregate growth rates over market value decreases in firm-level total factor productivity.

Table 2 – Decomposition of the variance of aggregate growth rates of productivity, sales, and profit. Table 2(a) reports the fraction of aggregate variance of productivity, sales, and profit growth rates that is accounted for by pairwise covariances in firm productivity, sales, and profit growth rates: $\sum_\omega \mathrm{Cov}(x, X)/\,\mathrm{Var}(X)$, where x and X are firm and aggregate growth rates. The fractions are reported as averages over years within each decade for the decades ending 1975, 1985, 1995, 2005, and 2015. Table 2(a) also reports summary statistics for each decade: average fraction of annual U.S. GDP accounted for by Compustat firms during, average number of active firms in Compustat, and total number of firm-year observations. The top panel of Table 2(b) reports the relative contributions that firms make to the variance of aggregate productivity, sales, and profit growth, for firms in different productivity deciles. The relative contributions are reported as averages for the firms within each productivity decile, relative to the cross-sectional average for firms in all deciles. The bottom panel of Table 2(b) shows the relative amounts of systematic risk that firms expose investor to.

(a) Sum of Pairwise Covariances, Relative to Aggregate Variance

Fraction of Variance	1966–1975	1976–1985	1986–1995	1996–2005	2006–2015
Productivity	0.81	0.85	0.90	0.84	0.85
s.e.	(0.03)	(0.03)	(0.01)	(0.03)	(0.04)
Sales	0.79	0.94	0.89	0.85	0.94
s.e.	(0.04)	(0.01)	(0.02)	(0.03)	(0.02)
Profit	0.69	0.85	0.88	0.83	0.72
s.e.	(0.05)	(0.01)	(0.01)	(0.05)	(0.17)
Decade Descriptions					
Avg Fraction of GDP	0.15	0.18	0.14	0.17	0.25
Avg Firms per Year	279	1026	1357	1519	1655
Firm-Year Observations	5481	13452	13291	15261	16951

(b) Firm-Aggregate Covariance, Firm-Aggregate Covariance Over Market Value, Relative to Average Firm

Covariance	Low	2	3	4	5	6	7	8	9	High
Productivity	0.02	0.06	0.06	0.11	0.14	0.26	0.46	0.84	1.63	6.43
s.e.	(0.01)	(0.00)	(0.01)	(0.01)	(0.01)	(0.05)	(0.02)	(0.10)	(0.14)	(0.27)
Sales	0.03	0.05	0.09	0.13	0.21	0.28	0.52	0.91	1.70	6.12
s.e.	(0.00)	(0.00)	(0.01)	(0.01)	(0.01)	(0.01)	(0.02)	(0.04)	(0.09)	(0.22)
Profit	0.01	0.02	0.04	0.06	0.11	0.18	0.45	0.85	1.46	6.85
s.e.	(0.00)	(0.00)	(0.01)	(0.01)	(0.01)	(0.02)	(0.04)	(0.05)	(0.09)	(0.29)
Covariance-Over-Value										
Productivity	1.97	2.53	1.07	1.22	0.77	0.67	0.51	0.32	0.09	0.35
s.e.	(1.29)	(0.35)	(0.19)	(0.11)	(0.10)	(0.08)	(0.11)	(0.19)	(0.03)	(0.06)
Sales	3.14	1.62	1.19	0.93	0.67	0.53	0.34	0.22	0.13	0.28
s.e.	(0.69)	(0.35)	(0.21)	(0.12)	(0.13)	(0.09)	(0.04)	(0.05)	(0.06)	(0.05)
Profit	1.43	3.64	1.73	0.81	0.79	0.61	0.34	0.27	0.10	0.28
s.e.	(1.65)	(0.47)	(0.21)	(0.16)	(0.12)	(0.10)	(0.06)	(0.04)	(0.04)	(0.06)
Decile Descriptions										
Avg Fraction of GDP	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Avg Firms per Year	1644	1908	1903	1863	1769	1587	1400	1159	891	701
Firm-Year Observations	6465	6442	6444	6444	6435	6451	6448	6440	6446	6421

The question is whether firm-level total factor productivity accounts for a non-negligible amount of the cross-sectional variation in firm-aggregate covariances and firm-aggregate covariances over value for productivity, sales, and profit in the Compustat data, after controlling for other sources of firm-aggregate covariance: the financial strength of firms, and firm characteristics like industry, age, and size. I find tentative support for the model's two predictions. Results are reported in Table 3.

The regression equation is given by

$$\frac{s_\omega \operatorname{Cov}(x_\omega, X)}{\text{Base}} = \beta_0 + \beta_1 \begin{pmatrix} \text{Total Factor} \\ \text{Productivity}_\omega \end{pmatrix} + \beta_2 \begin{pmatrix} \text{Financial} \\ \text{Strength}_\omega \end{pmatrix} + \beta_3 \begin{pmatrix} \text{Other} \\ \text{Controls}_{\omega,\Omega} \end{pmatrix} + \epsilon_\omega, \quad (30)$$

where x_ω is firm-level productivity, sales, or profit growth, X is firm-level productivity, sales, or profit growth, and where "Base" is either aggregate variance $\operatorname{Var}(X)$ or firm value V_ω.

The equation constitutes an estimated dependent variable model, because the left-hand side covariances are rolling-window estimates. Hausman (2001) and Lewis and Linzer (2005) discuss the econometric issues that arise in estimated dependent variable models like these. Hausman reminds us that an estimated dependent variable doesn't bias results as long as the classical ordinary least squares assumptions are met. Of course, they may not be met: Of particular concern is whether the firm covariances have sampling errors that vary in the cross-section of firms. Lewis and Linzer find that feasible generalized least squares works well if heteroskedasticity in the standard errors of the estimated dependent variable are large; otherwise, they recommend ordinary least squares using White's estimator (1980) for consistency. The right-hand side variables in Equation (30) include controls for sources of firm-level covariance that have been emphasized in previous studies. These alternative sources of firm-level covariance are briefly described below.

Financial Strength. Firms that rely on funds from banks and financial markets to run their businesses may respond similarly when the cost or availability of funds changes. Fama and French (1992) and Gertler and Gilchrist (1994) have argued that changes in access to external funds most affect small and financially weak firms, but recent work by Chari et al. (2007) and Crouzet and Mehrotra (2017) calls this view into question. I use firm leverage and liquidity ratios to control for financial strength as a source of comovement in Equation (30). For leverage, I follow Rajan and Zingales (1995) in defining three different ratios, the primary one being debt-to-equity. I follow Davydenko (2012) for measures of liquidity: cash and accounts receivable over current liabilities, and the interest coverage ratio as an alternative measure. Crouzet and Mehrotra (2017) use cash to assets.

Other controls. Firm fixed effects capture the impact of industry and other time-invariant firm-specific characteristics on firm-aggregate covariance. Industry effects might arise for a few reasons: First, industry-specific shocks generate higher covariance between firm pairs within an industry in the obvious way. Second, network effects may generate pairwise covariances. A shock to an individual industry can propagate outward from that industry to "connected" industries through the input-output network, where the propagation may run from supplier to customer (Acemoglu et al., 2012) or from customer to supplier (Herskovic et al., 2017). Unfortunately, these network connections are not captured by Compustat, so network effects not captured by firm fixed effects will show up in the error term. While the firm-level fixed effects control for time-invariant firm-level characteristics, I also explicitly control for two time-varying firm characteristics: size, and age. I control for these traits because large firms, and more mature firms, are known to differ systematically from their smaller, younger counterparts.

Table 3 reports regression results. The results lend tentative support to hypothesis (1) for productivity, sales, and profit growth, with positive coefficients on total factor productivity that are significant at the 10% level. In addition, larger firms (by employment) covary more with aggregate growth rates. Neither the financial control variables for leverage and liquidity

Table 3 – **Regressions of** $\text{Cov}(x, X)/Base$ **on firm productivity and control variables.** Results from the regression of $\text{Cov}(x, X)/Base$ on firm productivity and the following additional explanatory variables, where x and X are firm and aggregate growth rates, respectively, of total factor productivity, sales, and profit, and where $Base$ is either aggregate variance $\text{Var}(X)$ or firm market value $V_t(\omega)$. The ratio $\text{Cov}(x, X)/\text{Var}(X)$ represents each firm's contribution to aggregate variance as a share of the total, and the ratio $\text{Cov}(x, X)/V_t(\omega)$ represents the amount of systematic risk investors accept for every dollar they invest in the firm. The explanatory variables are: total factor productivity as the main variable of interest, liquidity as measured by the quick ratio, leverage as measured by the debt-to-capital ratio, firm age as measured by years in Compustat, firm size as measured by employment share. Regressions include firm fixed effects to capture unobserved time-invariant firm-level characteristics. Table (a) reports results from the regression of $\text{Cov}(x, X)/\text{Var}(X)$ on firm productivity and explanatory variables. Table (b) reports results from the regression of $\text{Cov}(x, X)/V_t(\omega)$ on firm productivity and explanatory variables.

(a) Regression of $\text{Cov}(x, X)/\text{Var}(X)$ **on firm productivity and control variables**

	x, X = Productivity Growth	Sales Growth	Profit Growth
Olley-Pakes Total Factor Productivity	0.053*	0.052*	0.171***
	(0.028)	(0.026)	(0.037)
Debt-to-Book Equity	0.006	0.002	0.004
	(0.005)	(0.004)	(0.004)
Quick Ratio	−0.000	−0.001	0.001
	(0.000)	(0.001)	(0.001)
Years in Compustat	0.022***	−0.005	−0.001
	(0.007)	(0.007)	(0.008)
Employment Share	0.124***	0.358***	0.253***
	(0.026)	(0.034)	(0.029)
R-squared	0.576	0.502	0.551

Robust standard errors in parentheses; * $p<0.10$, ** $p<0.05$, *** $p<0.01$

(b) Regression of $\text{Cov}(x, X)/V_t(\omega)$ **on firm productivity and control variables**

	x, X = Productivity Growth	Sales Growth	Profit Growth
Olley-Pakes Total Factor Productivity	−0.059***	−0.015**	−0.033***
	(0.013)	(0.008)	(0.012)
Debt-to-Book Equity	−0.018	0.005	0.003
	(0.027)	(0.013)	(0.010)
Quick Ratio	−0.008***	−0.015***	−0.011***
	(0.002)	(0.004)	(0.002)
Years in Compustat	−0.103***	−0.268***	−0.144***
	(0.007)	(0.008)	(0.007)
Employment Share	−0.000	0.012	0.035***
	(0.006)	(0.010)	(0.008)
R-squared	0.443	0.420	0.368

Robust standard errors in parentheses; * $p<0.10$, ** $p<0.05$, *** $p<0.01$

are significant. The results also lend support to hypothesis (2) for productivity, sales, and profit growth, with negative coefficients on total factor productivity that are significant at the 1% level for productivity and sales, and at the 5% level for profit. Leverage is insignificant, but higher liquidity appears to reduce firm-aggregate covariance over market value, as one might expect. Unfortunately, direct tests of firm-level diversification on firm-aggregate covariance using Compustat segment counts yielded insignificant results—likely due to the coarseness of the Compustat diversification measure (see Villalonga, 2004, for a discussion). More sophisticated methods of measuring the cross-sectional covariance structure, along with more granular diversification measures, will help test the diversification hypothesis more directly.

6 Conclusion

This paper advances the idea that when many firms choose similar risks, their economic fortunes then rise and fall together, creating aggregate fluctuations and systemic risks. To motivate this interpretation, I document four patterns in the comovement of firm-level productivity, sales, and profit growth rates for a large panel of public firms in the United States over the last half-century, and develop a model economy that produces the patterns endogenously. My contributions build on recent work on the microeconomic origins of aggregate fluctuations, on financial risk in production economies, and on endogenous fluctuations in macroeconomic models.

Empirical evidence from Compustat highlights the pervasiveness of firm-level covariance in all stages of value creation, and in most years during the last half-century. Figure 1 illustrates the evidence for productivity, sales, and profit growth rates, and Table 2 reports numerical results. Pairwise covariances in firm-level growth rates drive the variance of aggregate growth rates for these variables, in most years accounting for upwards of 80% of the aggregate variance. High-productivity firms are particularly important in generating the covariance in firm-level growth rates, contributing over six times what the average firm contributes. Despite the scale of their contributions to aggregate variance, high-productivity firms are less risky to investors per dollar of market value. They are less risky because they engage in at least some economic activities—operating certain technologies, selling to certain customers—that few other firms engage in.

The theoretical framework produces firm-level productivity covariance endogenously. When firms choose technology sets, they often choose to operate similar technologies. They are exposed to similar sources of technological uncertainty, and their productivities covary. The framework endogenously generates aggregate uncertainty from microeconomic shocks, captures qualitative features of the cross-section of stock returns, matches a host of stylized facts related to firm-level variance and covariance, and captures empirical regularities related to the number of technologies, product lines, and business segments firms operate. Yet the model remains highly tractable, and relies on a simple mechanism: firms choose their technologies, and therefore choose their risks.

Regressions serve as a plausibility check on the model's predictions. I regress the rolling-window firm-aggregate covariances for productivity, sales, and profit growth on firm-level total factor productivity, controlling for some common explanations of covariance in the literature: firm financial strength, fixed firm characteristics like industry, and time-varying ones like size and age. The regressions provide tentative support for the model's main predictions.

I see several new avenues of inquiry related to this work. The first is to examine cross-sectional covariance structure for small firms. In the present paper, I examine the upper tail of the firm size distribution, as Compustat excludes most small and all private firms. With access to administrative micro datasets, a similar exercise to this one could be carried out for small firms. Second: entry, exit, and dynamic aspects of the technology adoption and abandonment could be studied in an environment with endogenous uncertainty. When firms enter or exit, or change their technology sets, these activities can impact co-movement and volatility over time and in the cross-section in ways that could magnify aggregate fluctuations and systemic risk. Third: there is scope to more robustly characterize cross-sectional heterogeneity in firm-level productivity correlations than I have done here. Dynamic factor models and spatial econometric techniques offer two alternative approaches.

A Appendix

This appendix provides additional discussion on several aspects of the paper. It provides a basic mathematical discussion of the model presented in Section 3, proofs of the propositions in Section 4, details of the productivity estimation procedure I use. The discussion includes first-order conditions for the decision problems of the representative household and of consumption goods producers, and derivations of the propositions presented in section 4.

A.1 Mathematical discussion of the model

A.1.1 Optimality conditions

Optimality conditions for the representative household. The household solves its utility maximization problem in two stages. The two-stage budgeting procedure is possible here because the period utility function $u(C_s)$ depends only on the basket C_t, and C_t is homogeneous of degree one (Gorman, 1959). Consider the first-stage problem in (11). Eliminate constraint (10) by substituting for I_t in (9). Use the method of Lagrangian multipliers to rewrite the objective function as

$$\mathcal{L} = \mathrm{E}\left[\sum_{s=t}^{\infty} \beta^{s-t} u(C_s) - \beta^{s-t}\lambda_s \left(C_s + K_{s+1} + \int_{\omega\in\Omega} V_s(\omega) S_{s+1}(\omega)\lambda(d\omega) \right.\right.$$
$$\left.\left. - w_s L - (1 + r_s - \delta) K_s - \int_{\omega\in\Omega} [V_s(\omega) - \Pi_s(\omega)] S_s(\omega)\lambda(d\omega) \right)\right]. \quad (31)$$

To get first-order optimality conditions, equate with zero the first derivatives of $\mathcal{L}$ with respect to choice variables C_s, K_{s+1}, $S_{s+1}(\omega)$, and λ_s for arbitrary period s and firm ω. The household's optimal plans for consumption, capital accumulation, and equity shares, respectively, satisfy the following conditions:

$$\mathrm{E}\big[u'(C_s)\big] = \mathrm{E}[\lambda_s], \tag{32}$$

$$\mathrm{E}[\lambda_s] = \beta\mathrm{E}\big[\lambda_{s+1}(1 + r_{s+1} - \delta)\big], \tag{33}$$

$$\mathrm{E}\big[\lambda_s V_s(\omega)\big] = \beta\mathrm{E}\big[\lambda_{s+1}(V_{s+1}(\omega) + \Pi_{s+1}(\omega))\big]. \tag{34}$$

The household's stochastic discount factor also derives from these conditions: set $s = t$ and use (32) and (34) to write firm ω's period-t present value as

$$V_t(\omega) = \mathrm{E}_t\left[\left(\beta\frac{u'(C_{t+1})}{u'(C_t)}\right)\left(V_{t+1}(\omega) + \Pi_{t+1}(\omega)\right)\right]. \tag{35}$$

The one-period stochastic discount factor is then the first term in the expectation operator: $m_{t,t+1} = \beta u'(C_{t+1})/u'(C_t)$. Iterate (35) via $V_{t+1}(\omega)$ to get the multi-period stochastic discount factor. For any period $s \geq t$, write the latter as

$$\begin{aligned} m_{t,s} &= m_{t,t+1} \cdot m_{t+1,t+2} \cdots m_{s-1,s} \\ &= \beta\frac{u'(C_{t+1})}{u'(C_t)} \cdot \beta\frac{u'(C_{t+2})}{u'(C_{t+1})} \cdots \beta\frac{u'(C_s)}{u'(C_{s-1})} = \beta^{s-t}\frac{u'(C_s)}{u'(C_t)}. \end{aligned} \tag{36}$$

Next, solve the household's second-stage problem of allocating consumption across varieties $c_t(v,\omega)$ within the aggregate basket C_t. Let $P_t(v,\omega)$ be the nominal price of variety $c_t(v,\omega)$, and P_t be the nominal price of the consumption basket C_t. The household takes the optimal amount of aggregate consumption C_t as given by the first-stage problem, and takes nominal prices as given, and maximizes its consumption of varieties for each unit of expenditure $1 := P_t C_t$, by solving equation (12). Writing the Lagrangian,

$$\mathcal{L} = \left[\int\limits_{\Omega} \int\limits_{\mathcal{V}(\omega)} \left[c_t(v,\omega)\right]^{\frac{\theta-1}{\theta}} \lambda(dvd\omega) \right]^{\frac{\theta}{\theta-1}} + \lambda_t \left[1 - \int\limits_{\Omega} \int\limits_{\mathcal{V}(\omega)} P_t(v,\omega) c_t(v,\omega) \lambda(dvd\omega) \right].$$

Taking the first derivative of the Lagrangian with respect to consumption varieties $c_t(v,\omega), c_t(v',\omega')$, and setting equal to zero,

$$C_t^{-1} c_t(v,\omega)^{-\frac{1}{\theta}} = P_t(v,\omega),$$
$$C_t^{-1} c_t(v',\omega')^{-\frac{1}{\theta}} = P_t(v',\omega'),$$

and the ratio of the two optimality conditions yields,

$$\left(\frac{c_t(v,\omega)}{c_t(v',\omega')} \right)^{-\frac{1}{\theta}} = \frac{P_t(v,\omega)}{P_t(v',\omega')}. \tag{37}$$

Using equation (37) in the expenditure constraint in equation (12),

$$\begin{aligned} 1 &= \int\limits_{\Omega} \int\limits_{\mathcal{V}(\omega)} P_t(v,\omega) c_t(v,\omega) \lambda(dvd\omega) \\ &= \int\limits_{\Omega} \int\limits_{\mathcal{V}(\omega)} P_t(v,\omega) \left(\frac{P_t(v',\omega')}{P_t(v,\omega)} \right)^{\theta} c_t(v',\omega') \lambda(dvd\omega) \\ &= P_t(v',\omega')^{\theta} c_t(v',\omega') \int\limits_{\Omega} \int\limits_{\mathcal{V}(\omega)} P_t(v,\omega)^{1-\theta} \lambda(dvd\omega). \end{aligned}$$

Again using equation (37), notice that the aggregate consumption basket can be written

$$\begin{aligned}
C_t &= \left[\int\limits_{\Omega} \int\limits_{\mathcal{V}(\omega)} \left[c_t(v,\omega)\right]^{\frac{\theta-1}{\theta}} \lambda(dv d\omega) \right]^{\frac{\theta}{\theta-1}} \\
&= \left[\int\limits_{\Omega} \int\limits_{\mathcal{V}(\omega)} \left[\left(\frac{P_t(v,\omega)}{P_t(v',\omega')} \right)^{-\theta} c_t(v',\omega') \right]^{\frac{\theta-1}{\theta}} \lambda(dv d\omega) \right]^{\frac{\theta}{\theta-1}} \\
&= P_t(v',\omega')^{\theta} c_t(v',\omega') \left[\int\limits_{\Omega} \int\limits_{\mathcal{V}(\omega)} P_t(v,\omega)^{1-\theta} \lambda(dv d\omega) \right]^{\frac{\theta}{\theta-1}}.
\end{aligned}$$

Now recall $1 = P_t C_t$, and define $p_t(v,\omega) := P_t(v,\omega)/P_t$. The above expressions imply the following price index and demand curve:

$$1 = \left[\int\limits_{\Omega} \int\limits_{\mathcal{V}(\omega)} \left[p_t(v,\omega)\right]^{1-\theta} \lambda(dv d\omega) \right]^{\frac{1}{1-\theta}}, \tag{38}$$

$$c_t(v,\omega) = \left[p_t(v,\omega)\right]^{-\theta} C_t. \tag{39}$$

Optimality conditions for consumption goods producers. Consider firm ω's profit maximization problem (5). Eliminate constraints by using (3) and (39) to substitute for $p_t(v,\omega)$ and $y_t(v,\omega)$ in the firm-vintage profit function (4) that appears in (5). Obtain first-order optimality conditions by equating with zero the first derivatives of $\Pi_t(\omega)$ with respect to choice variables $k_t(v,\omega)$ and $l_t(v,\omega)$ for arbitrary vintage v. Firm ω's optimal choice of capital for production with vintage v satisfies

$$k_t(v,\omega) = (\alpha)\left(\frac{\theta-1}{\theta}\right)(Y_t)^{\frac{1}{\theta}}[y_t(v,\omega)]^{\frac{\theta-1}{\theta}}(r_t)^{-1}. \tag{40}$$

Its optimal choice of labor satisfies

$$l_t(v,\omega) = (1-\alpha)\left(\frac{\theta-1}{\theta}\right)(Y_t)^{\frac{1}{\theta}}[y_t(v,\omega)]^{\frac{\theta-1}{\theta}}(w_t)^{-1}. \tag{41}$$

Notice that the optimal capital-labor ratio depends neither on the individual firm nor on the vintage of technology:

$$\frac{k_t(v,\omega)}{l_t(v,\omega)} = \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{w_t}{r_t}\right). \tag{42}$$

Optimality conditions for capital goods producers. Now consider the profit maximization problem in the (13). Take derivatives of gross profit with respect to the factors to obtain first-order conditions:

$$r_t = \alpha Z_t(k_t)^{\alpha-1}(l_t)^{1-\alpha}, \tag{43}$$

$$w_t = (1-\alpha)Z_t(k_t)^{\alpha}(l_t)^{-\alpha}. \tag{44}$$

Notice that the the capital-labor ratio in the capital goods sector is again

$$\frac{k_t}{l_t} = \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{w_t}{r_t}\right). \tag{45}$$

A.1.2 Main propositions and proofs

Proposition 4.1 *Let $z_t(v)$ now be a random preference multiplier. Replace the stochastic production function in equation (3) with equation (3′) below, and the non-stochastic preferences in equation (12) with equation (12′) below:*

$$y_t(v,\omega) = z(\omega)\big[k_t(v,\omega)\big]^{\alpha}\big[l_t(v,\omega)\big]^{1-\alpha}, \tag{3′}$$

$$C_t = \left[\int\limits_{\Omega}\int\limits_{\mathcal{V}(\omega)} \big[z_t(v)c_t(v,\omega)\big]^{\frac{\theta-1}{\theta}}\lambda(dvd\omega)\right]^{\frac{\theta}{\theta-1}}. \tag{12′}$$

Then the propositions of this section remain true after derivation of the appropriate stochastic household demand curve for individual varieties.

Proof. The proof starts by re-deriving the demand-curve, following appendix A.1.1 but now under the stochastic preferences in equation (12′). The household solves

$$\begin{aligned}
&\max_{\{c_t(v,\omega)\}_{v\in\mathcal{V},\omega\in\Omega}} && \left[\int\limits_{\Omega}\int\limits_{\mathcal{V}(\omega)} \big[z_t(v)c_t(v,\omega)\big]^{\frac{\theta-1}{\theta}}\lambda(dvd\omega)\right]^{\frac{\theta}{\theta-1}} \\
&\text{s.t.} && 1 = \int\limits_{\Omega}\int\limits_{\mathcal{V}(\omega)} P_t(v,\omega)c_t(v,\omega)\lambda(dvd\omega).
\end{aligned} \tag{46}$$

The Lagrangian is

$$\mathcal{L} = \left[\int\limits_{\Omega}\int\limits_{\mathcal{V}(\omega)} \big[z_t(v)c_t(v,\omega)\big]^{\frac{\theta-1}{\theta}}\lambda(dvd\omega)\right]^{\frac{\theta}{\theta-1}} + \lambda_t\left[1 - \int\limits_{\Omega}\int\limits_{\mathcal{V}(\omega)} P_t(v,\omega)c_t(v,\omega)\lambda(dvd\omega)\right].$$

Taking the first derivative of the Lagrangian with respect to consumption varieties $c_t(v, \omega), c_t(v', \omega')$, and setting equal to zero,

$$C_t^{-1}\left[z_t(v)c_t(v,\omega)\right]^{-\frac{1}{\theta}} z_t(v) = P_t(v,\omega),$$
$$C_t^{-1}\left[z_t(v')c_t(v',\omega')\right]^{-\frac{1}{\theta}} z_t(v') = P_t(v',\omega'),$$

and the ratio of the two optimality conditions yields,

$$\left(\frac{z_t(v)}{z_t(v')}\right)^{\frac{\theta-1}{\theta}} \left(\frac{c_t(v,\omega)}{c_t(v',\omega')}\right)^{-\frac{1}{\theta}} = \frac{P_t(v,\omega)}{P_t(v',\omega')}. \tag{47}$$

The remaining steps of the derivation are straight-forward and follow the derivation in appendix A.1.1 closely. □

Proposition 4.2 *A productivity aggregate over technologies summarizes all of the technological heterogeneity within an individual firm* ω*:*

$$Z_t(\omega) = \left[\int_{\mathcal{V}(\omega)} \left[z(\omega)z_t(v)\right]^{\theta-1}\lambda(dv)\right]^{\frac{1}{\theta-1}}. \tag{48}$$

A productivity aggregate over firms summarizes all of the firm-specific and technological heterogeneity within the consumption goods sector:

$$Z_t = \left[\int_{\Omega} Z_t(\omega)^{\theta-1}\lambda(d\omega)\right]^{\frac{1}{\theta-1}}. \tag{49}$$

Aggregate factor demands, production, and profit can be written in terms of aggregate productivities and variables that either do not vary across firms, in the case of firm aggregates, *or do not vary across firms or technologies, in the case of economy-wide aggregates.*

Proof. The household and capital goods producer are representative, so aggregation pertains only to the final goods sector.

Start with the optimality conditions (40) and (41) from the firm's decision problem (5). These expressions contain vintage-specific variables $k_t(v, \omega)$, $l_t(v, \omega)$, and $y_t(v, \omega)$ as well as variables and parameters common to all vintages. Combine equations (40) and (41) with the production function (3) to obtain expressions for $k_t(v, \omega)$, $l_t(v, \omega)$, and $y_t(v, \omega)$ in terms of $z_t(v)$ and variables and parameters common to all vintages:

$$k_t(v,\omega) = [z(\omega)z_t(v)]^{\theta-1}(Y_t)\left(\frac{\theta-1}{\theta}\right)^{\theta}\left(\frac{r_t}{\alpha}\right)^{\alpha(1-\theta)-1}\left(\frac{w_t}{1-\alpha}\right)^{(1-\alpha)(1-\theta)}, \tag{50}$$

$$l_t(v,\omega) = [z(\omega)z_t(v)]^{\theta-1}(Y_t)\left(\frac{\theta-1}{\theta}\right)^{\theta}\left(\frac{r_t}{\alpha}\right)^{\alpha(1-\theta)}\left(\frac{w_t}{1-\alpha}\right)^{(1-\alpha)(1-\theta)-1}, \tag{51}$$

$$y_t(v,\omega) = [z(\omega)z_t(v)]^{\theta}(Y_t)\left(\frac{\theta-1}{\theta}\right)^{\theta}\left(\frac{r_t}{\alpha}\right)^{-\alpha\theta}\left(\frac{w_t}{1-\alpha}\right)^{-(1-\alpha)\theta}. \tag{52}$$

These expressions can be simplified further using an expression derived from the definition of the consumption basket, along with (52) and market clearing:

$$\begin{aligned} Y_t &= \left[\int_{\Omega}\int_{\mathcal{V}(\omega)} [y_t(v,\omega)]^{\frac{\theta-1}{\theta}}\lambda(dvd\omega)\right]^{\frac{\theta}{\theta-1}} \\ &= \left(\frac{\theta-1}{\theta}\right)^{\theta}\left(\frac{\alpha}{r_t}\right)^{\alpha\theta}\left(\frac{1-\alpha}{w_t}\right)^{(1-\alpha)\theta}(Y_t)\left[\int_{\Omega}\int_{\mathcal{V}(\omega)} (z(\omega)z_t(v))^{\theta-1}\lambda(dvd\omega)\right]^{\frac{\theta}{\theta-1}} \\ \Leftrightarrow \qquad Z_t &:= \left[\int_{\Omega}\int_{\mathcal{V}(\omega)} (z(\omega)z_t(v))^{\theta-1}\lambda(dvd\omega)\right]^{\frac{1}{\theta-1}} = \left(\frac{\theta}{\theta-1}\right)\left(\frac{r_t}{\alpha}\right)^{\alpha}\left(\frac{w_t}{1-\alpha}\right)^{1-\alpha}. \end{aligned}$$

Now use the expression for Z_t to simplify (50)–(52):

$$
\begin{aligned}
k_t(v,\omega) &= \left(\frac{\theta-1}{\theta}\right)\left(\frac{\alpha}{r_t}\right)\left(\frac{z(\omega)z_t(v)}{Z_t}\right)^{\theta-1} Y_t \\
l_t(v,\omega) &= \left(\frac{\theta-1}{\theta}\right)\left(\frac{1-\alpha}{w_t}\right)\left(\frac{z(\omega)z_t(v)}{Z_t}\right)^{\theta-1} Y_t \\
y_t(v,\omega) &= \left(\frac{z(\omega)z_t(v)}{Z_t}\right)^{\theta} Y_t.
\end{aligned}
$$

Now recall that $p_t(v,\omega) = (y_t(v,\omega)/Y_t)^{-(1/\theta)}$, and use above to get a similar expression for profit:

$$
\begin{aligned}
\pi_t(v,\omega) &= p_t(v,\omega)y_t(v,\omega) - r_t k_t(v,\omega) - w_t l_t(v,\omega) \\
&= \frac{1}{\theta}\left(\frac{z(\omega)z_t(v)}{Z_t}\right)^{\theta-1} Y_t.
\end{aligned}
$$

To get firm aggregates, sum the $k_t(v,\omega)$'s, $l_t(v,\omega)$'s, and $\pi_t(v,\omega)$'s, and use the Dixit-Stiglitz aggregator on $y_t(v,\omega)$:

$$
\begin{aligned}
K_t(\omega) &:= \int\limits_{\mathcal{V}(\omega)} k_t(v,\omega)\lambda(dv) = \left(\frac{\theta-1}{\theta}\right)\left(\frac{\alpha}{r_t}\right)\left(\frac{Z_t(\omega)}{Z_t}\right)^{\theta-1} Y_t \\
L_t(\omega) &:= \int\limits_{\mathcal{V}(\omega)} l_t(v,\omega)\lambda(dv) = \left(\frac{\theta-1}{\theta}\right)\left(\frac{1-\alpha}{w_t}\right)\left(\frac{Z_t(\omega)}{Z_t}\right)^{\theta-1} Y_t \\
Y_t(\omega) &:= \left[\int\limits_{\mathcal{V}(\omega)} (y_t(v,\omega))^{\frac{\theta-1}{\theta}}\lambda(dv)\right]^{\frac{\theta}{\theta-1}} = \left(\frac{Z_t(\omega)}{Z_t}\right)^{\theta} Y_t \\
\Pi_t(\omega) &:= \int\limits_{\mathcal{V}(\omega)} \pi_t(v,\omega)\lambda(dv) = \frac{1}{\theta}\left(\frac{Z_t(\omega)}{Z_t}\right)^{\theta-1} Y_t,
\end{aligned}
$$

where

$$Z_t(\omega) := \left[\int\limits_{\mathcal{V}(\omega)} (z(\omega) z_t(v))^{\theta-1} \lambda(dv) \right]^{\frac{1}{\theta-1}}.$$

Further rearrangement along the same lines yields the economy-wide aggregates. It is also possible to write aggregate output in terms of a Cobb-Douglas aggregate production function, at both the firm and economy-wide levels:

$$Y_t(\omega) = Z_t(\omega)[K_t(\omega)]^{\alpha}[L_t(\omega)]^{1-\alpha} \tag{53}$$

$$Y_t = Z_t(K_t)^{\alpha}(L_t)^{1-\alpha}, \tag{54}$$

where the production function arguments should be understood as *optimal* factor inputs that satisfy the firm's optimality conditions for from the profit maximization problem (see Felipe and Fisher, 2003, for a discussion).

Notice that the firm-level aggregate production function takes the familiar Cobb-Douglas form. But remember that the distribution of shocks is endogenous, and the underlying technology choice problem imposes additional structure on the firm-level productivity multipliers. In particular, if technology sets $\mathcal{V}(\omega)$ differs across firms, so too will the distributions of the random productivity multipliers. And to the extent that technology sets share common elements, firm-level productivity will covary. The next three propositions make these statements rigorous. □

Proposition 4.3 *In non-stochastic steady state, any firm* ω *with productivity* $z(\omega) \geq \underline{z}$ *chooses technology set* $\mathcal{V}(\omega) = \left\{ v \in \mathcal{V} : \underline{v} \leq v \leq \overline{v}(\omega) \right\}$, *where the endogenous cut-offs* $\underline{z}$ *and*

$\overline{v}(\omega)$ *are given by:*

$$\underline{z} = \left(\frac{\theta}{\mu_\epsilon}\right)^{\frac{1}{\theta-1}} \tag{55}$$

$$\overline{v}(\omega) = \left(\frac{\mu_\epsilon}{\theta}\right)^{\frac{1}{\gamma}} z(\omega)^{\frac{\theta-1}{\gamma}}. \tag{56}$$

Firms with $z(\omega) < \underline{z}$ *do not produce. Under parameter restrictions, firms* ω_1 *and* ω_2 *with productivities* $\underline{z} < z(\omega_1) < z(\omega_2)$ *choose technology sets such that* $\mathcal{V}_t(\omega_1) \subset \mathcal{V}_t(\omega_2)$. *The above cut-offs are also first-order approximate to those that obtain in a stochastic environment.*

Proof. Firms choose their technology sets $\mathcal{V}(\omega) \subseteq \mathcal{V} = [\underline{v}, \infty) \subseteq \mathbb{R}^+$ to maximize profit. Recall from subsubsection 3.1.3 that technologies differ in their period fixed costs, but not their first two moments. Starting from the technology adoption rule in (7), and rearranging:

$$\begin{aligned}
0 &< \mathrm{E}_t\left[m_{t,t+1}(\pi_t(v,\omega) - f_s(v))\right] \\
&= \mathrm{E}_t\left[\beta\frac{u'(C_{t+1})}{u'(C_t)}(\pi_t(v,\omega) - f_s(v))\right] \\
&= \mathrm{E}_t\left[\frac{1}{Y_{t+1}}(\pi_{t+1}(v,\omega) - f_{t+1}(v))\right],
\end{aligned}$$

where the third line assumes log utility. Now recall from the proof to Proposition 4.2:

$$\begin{aligned}
\pi_t(v,\omega) &= \frac{1}{\theta}\left(\frac{z(\omega)z_t(v)}{Z_t}\right)^{\theta-1} Y_t, \\
f_t(v) &= \frac{Y_t}{\mu}v^\gamma.
\end{aligned}$$

Using these expressions in the adoption rule:

$$
\begin{aligned}
& \mathrm{E}_t\left[\frac{1}{Y_{t+1}}(\pi_{t+1}(v,\omega) - f_{t+1}(v))\right] > 0 \\
\Leftrightarrow \qquad & \left(\frac{z(\omega)^{\theta-1}}{\theta}\right)\mathrm{E}_t\left[\left(\frac{z_t(v)}{Z_t}\right)^{\theta-1}\right] \geq \frac{v^\gamma}{\mu}.
\end{aligned}
$$

From here, either evaluate the productivities in the ratio under the expectation operator at their expected values to get an expression describing steady-state technology sets, or take an approximation of the expression under the expectation operator. A first-order approximation gives the same results as the steady-state solution:

$$
\overline{v}(\omega) = \left(\frac{\mu_\epsilon}{\theta}\right)^{\frac{1}{\gamma}} z(\omega)^{\frac{\theta-1}{\gamma}}
$$

$$
\Rightarrow \qquad \underline{z}(v) = \left(\frac{\mu_\epsilon}{\theta}\right)^{\frac{1}{\theta-1}} v^{\frac{\gamma}{\theta-1}}.
$$

Notice that the cut-off $\overline{v}(\omega)$ increasing in $z(\omega)$, so the more productive firms produce more varieties and use more technology.

Two remarks are in order: First, it is useful that the steady-state and first-order approximate cut-offs coincide, because it means that first-order dynamics around the steady state are completely standard in this model. Second, the second-order approximate case gives more interesting but less tractable results. There is a covariance term in the second-oder approximation that varies with v—covariance is higher for commonly-used technologies. $\square$

Proposition 4.4 *Let technology sets be those that firms choose in non-stochastic steady state. Then the first and second moments of firm-level productivity are given by* $\mu(\omega)$ *and*

$\sigma^2(\omega)$, *respectively:*

$$\mu(\omega) = \mu_\epsilon z(\omega)^{\zeta_{\mu\omega 1}} \left[\left(\frac{z(\omega)}{\underline{z}} \right)^{\zeta_{\mu\omega 2}} - 1 \right], \tag{57}$$

$$\sigma^2(\omega) = \sigma_\epsilon^2 z(\omega)^{\zeta_{\sigma\omega 1}} \left[\left(\frac{z(\omega)}{\underline{z}} \right)^{\zeta_{\sigma\omega 2}} - 1 \right]. \tag{58}$$

The first and second moments of aggregate productivity are given by μ *and* σ^2*, respectively:*

$$\mu = \mu_\epsilon \zeta_{\mu 1} \underline{z}^{\zeta_{\mu 2}}, \tag{59}$$

$$\sigma^2 = \sigma_\epsilon^2 \zeta_{\sigma 1} \underline{z}^{\zeta_{\sigma 2}}. \tag{60}$$

Under parameter restrictions, the first and second moments of all productivity aggregates are positive and finite, and for any firms ω_1 *and* ω_2 *with* $z(\omega_1) < z(\omega_2)$*, it holds that* $\mu_t(\omega_1) < \mu_t(\omega_2)$ *and* $\sigma_t^2(\omega_1) < \sigma_t^2(\omega_2)$*.*

Proof. Begin with the first moment of sector-aggregate productivity, just using the definition:

$$\mu = \mathrm{E}\Big[Z_t^{\theta-1}\Big] = \mathrm{E}\left[\int\limits_{\Omega} Z_t(\omega)^{\theta-1}\lambda(d\omega)\right]$$

$$= \mathrm{E}\left[\int\limits_{\Omega}\int\limits_{\mathcal{V}(\omega)} \big(z(\omega)z_t(v)\big)^{\theta-1}\lambda(dvd\omega)\right]$$

$$= \mathrm{E}\left[\int\limits_{\mathcal{V}}\int\limits_{\Omega_v} \big(z(\omega)z_t(v)\big)^{\theta-1}\lambda(d\omega dv)\right]$$

$$= \mathrm{E}\left[\int\limits_{\mathcal{V}} z_t(v)^{\theta-1}\left(\int\limits_{\Omega_v} z(\omega)^{\theta-1}\lambda(d\omega)\right)\lambda(d\omega)\right],$$

where Ω_v is the set of firms using vintage v, that is: $\Omega_v := \big\{\omega \in \Omega : \underline{z}(v) < z(\omega)\big\}$, and $\underline{z}(v)$ is the inverse of the cost cut-off $\overline{v}(\omega)$.

Now evaluate the inner integral:

$$\int\limits_{\Omega_v} z(\omega)^{\theta-1}\lambda(d\omega) = \int\limits_{\underline{z}(v)}^{\infty} z(\omega)^{\theta-1}h(z(\omega))dz(\omega)$$

$$= \left[\frac{\kappa}{(\theta-1)-\kappa} z(\omega)^{(\theta-1)-\kappa}\right]_{\underline{z}(v)}^{\infty}$$

$$= \left(\frac{\kappa}{\kappa-(\theta-1)}\right)\underline{z}(v)^{(\theta-1)-\kappa}$$

$$= \left(\frac{\kappa}{\kappa-(\theta-1)}\right)\left(\frac{\mu_\epsilon}{\theta}\right)^{\frac{\kappa-(\theta-1)}{\theta-1}}\left(\frac{1}{v}\right)^{\frac{\gamma[\kappa-(\theta-1)]}{\theta-1}}.$$

Substitute the evaluated integral back into the expression for μ:

$$\begin{aligned}\mu = \mathrm{E}\Big[Z_t^{\theta-1}\Big] &= \left(\frac{\kappa}{\kappa-(\theta-1)}\right)\left(\frac{\mu_\epsilon}{\theta}\right)^{\frac{\kappa-(\theta-1)}{\theta-1}} \mathrm{E}\left[\int\limits_{\mathcal{V}} z_t(v)^{\theta-1} v^{-\frac{\gamma[\kappa-(\theta-1)]}{\theta-1}} \lambda(dv)\right] \\ &= \left(\frac{\kappa}{\kappa-(\theta-1)}\right)\left(\frac{\mu_\epsilon}{\theta}\right)^{\frac{\kappa-(\theta-1)}{\theta-1}} \mathrm{E}\left[\int\limits_{\underline{v}}^{\infty} z_t(v)^{\theta-1} v^{-\frac{\gamma[\kappa-(\theta-1)]}{\theta-1}} \lambda(dv)\right]\end{aligned}$$

Use the definition of technological productivity $z_t(v) := \epsilon_{t,\lceil v\rceil}$, set $\underline{v} = 1$, and write the remaining integral as:

$$\begin{aligned}\mathrm{E}\left[\int\limits_{\underline{v}}^{\infty} z_t(v)^{\theta-1} v^{-\frac{\gamma[\kappa-(\theta-1)]}{\theta-1}} \lambda(dv)\right] &= \mathrm{E}\left[\int\limits_{\underline{v}}^{\infty} \epsilon_{t,\lceil v\rceil}^{\theta-1} v^{-\frac{\gamma[\kappa-(\theta-1)]}{\theta-1}} \lambda(dv)\right] \\ &= \mathrm{E}\left[\int\limits_{1}^{2} \epsilon_{t,2}^{\theta-1} v^{-\frac{\gamma[\kappa-(\theta-1)]}{\theta-1}} \lambda(dv) + \int\limits_{2}^{3} \epsilon_{t,3}^{\theta-1} v^{-\frac{\gamma[\kappa-(\theta-1)]}{\theta-1}} \lambda(dv) + \dots\right] \\ &= \mu_\epsilon \int\limits_{1}^{2} v^{-\frac{\gamma[\kappa-(\theta-1)]}{\theta-1}} \lambda(dv) + \mu_\epsilon \int\limits_{2}^{3} v^{-\frac{\gamma[\kappa-(\theta-1)]}{\theta-1}} \lambda(dv) + \dots.\end{aligned}$$

Now consider the integrals of the form:

$$\begin{aligned}\int\limits_{n}^{n+1} v^{-\frac{\gamma[\kappa-(\theta-1)]}{\theta-1}} \lambda(dv) &= \left[\left(\frac{\theta-1}{\gamma[\kappa-(\theta-1)]+(\theta-1)}\right) v^{\frac{-\gamma[\kappa-(\theta-1)]+(\theta-1)}{\theta-1}}\right]_n^{n+1} \\ &= \left(\frac{\theta-1}{\gamma[\kappa-(\theta-1)]+(\theta-1)}\right)\left[\left(\frac{1}{n}\right)^{\frac{\gamma[\kappa-(\theta-1)]+(\theta-1)}{\theta-1}} - \left(\frac{1}{n+1}\right)^{\frac{\gamma[\kappa-(\theta-1)]+(\theta-1)}{\theta-1}}\right].\end{aligned}$$

Returning to the expression for μ:

$$\begin{aligned}
\mu = \mathrm{E}\Big[Z_t^{\theta-1}\Big] &= \left(\frac{\kappa}{\kappa-(\theta-1)}\right)\left(\frac{\mu_\epsilon}{\theta}\right)^{\frac{\kappa-(\theta-1)}{\theta-1}} \mu_\epsilon \sum_{n=1}^{\infty}\left(\frac{\theta-1}{\gamma[\kappa-(\theta-1)]-(\theta-1)}\right) \\
&\qquad \times \left[\left(\frac{1}{n}\right)^{\frac{\gamma[\kappa-(\theta-1)]+(\theta-1)}{\theta-1}} - \left(\frac{1}{n+1}\right)^{\frac{\gamma[\kappa-(\theta-1)]+(\theta-1)}{\theta-1}}\right] \\
&= \mu_\epsilon\left(\frac{\theta-1}{\gamma[\kappa-(\theta-1)]-(\theta-1)}\right)\left(\frac{\kappa}{\kappa-(\theta-1)}\right)\left(\frac{\mu_\epsilon}{\theta}\right)^{\frac{\kappa-(\theta-1)}{\theta-1}}.
\end{aligned}$$

Notice that $\left(\frac{\mu_\epsilon}{\theta}\right)^{\frac{1}{\theta-1}}$ appears on the right-hand side. Substituting it for $\underline{z}$, and collecting parameters,

$$\mu = \mu_\epsilon \zeta_{\mu 1} \underline{z}^{\zeta_{\mu 2}},$$

where

$$\begin{aligned}
\zeta_{\mu 1} &:= \left(\frac{\theta-1}{\gamma[\kappa-(\theta-1)]-(\theta-1)}\right)\left(\frac{\kappa}{\kappa-(\theta-1)}\right) \\
\zeta_{\mu 2} &:= \kappa-(\theta-1)
\end{aligned}$$

Now turn to the second moment of sector-aggregate productivity. Starting again with the definition:

$$\begin{aligned}
\sigma^2 = \operatorname{Var}\left(Z_t^{\theta-1}\right) &= \operatorname{Var}\left(\int_{\Omega} Z_t(\omega)^{\theta-1}\lambda(d\omega)\right) \\
&= \operatorname{Var}\left(\int_{\Omega}\int_{\mathcal{V}(\omega)} (z(\omega)z_t(v))^{\theta-1}\lambda(dvd\omega)\right) \\
&= \operatorname{Var}\left(\int_{\mathcal{V}} z_t(v)^{\theta-1}\int_{\Omega_v} z(\omega)^{\theta-1}\lambda(d\omega dv)\right) \\
&= \operatorname{Var}\left(\int_{\mathcal{V}} z_t(v)^{\theta-1}\int_{\underline{z}(v)}^{\infty} z(\omega)^{\theta-1}\frac{\kappa}{z(\omega)^{\kappa+1}}\lambda(dz(\omega)dv)\right) \\
&= \operatorname{Var}\left(\int_{\mathcal{V}} z_t(v)^{\theta-1}\frac{\kappa}{(\theta-1)-\kappa}\underline{z}(v)^{-[\kappa-(\theta-1)]}\lambda(d\omega)\right),
\end{aligned}$$

where from the third to the fourth line I change measure from Lebesgue to Pareto. Continuing, using $\underline{z}(v) = \left(\frac{\theta}{\mu_\epsilon}\right)^{\frac{1}{\theta-1}} v^{\frac{\gamma}{\theta-1}}$,

$$\begin{aligned}
\sigma^2 = \operatorname{Var}\left(Z_t^{\theta-1}\right) &= \operatorname{Var}\left(\int_{\mathcal{V}}\left(\frac{\kappa}{(\theta-1)-\kappa}\right)\left(\frac{\theta}{\mu_\epsilon}\right)^{-\frac{\kappa-(\theta-1)}{\theta-1}} z_t(v)^{\theta-1} v^{\frac{-\gamma[\kappa-(\theta-1)]}{\theta-1}}\lambda(d\omega)\right) \\
&= \left(\frac{\kappa}{\kappa-(\theta-1)}\right)^2\left(\frac{\theta}{\mu_\epsilon}\right)^{-2\frac{\kappa-(\theta-1)}{\theta-1}} \operatorname{Var}\left(\int_{\mathcal{V}} z_t(v)^{\theta-1} v^{-\frac{\gamma[\kappa-(\theta-1)]}{\theta-1}}\right).
\end{aligned}$$

Now consider the integral:

$$
\begin{aligned}
\int\limits_{\mathcal{V}} z_t(v)^{\theta-1} v^{-\frac{\gamma[\kappa-(\theta-1)]}{\theta-1}} \lambda(d\omega) &= \int\limits_{\mathcal{V}} \epsilon_{t,\lceil v \rceil} v^{-\frac{\gamma[\kappa-(\theta-1)]}{\theta-1}} \lambda(d\omega) \\
&= \int\limits_{\overline{v}(\omega)}^{\infty} \epsilon_{t,\lceil v \rceil} v^{-\frac{\gamma[\kappa-(\theta-1)]}{\theta-1}} \lambda(d\omega) \\
&= \epsilon_{t,2} \int\limits_{1}^{2} v^{-\frac{\gamma[\kappa-(\theta-1)]}{\theta-1}} \lambda(d\omega) + \epsilon_{t,3} \int\limits_{2}^{3} v^{-\frac{\gamma[\kappa-(\theta-1)]}{\theta-1}} \lambda(d\omega) + \ldots \\
&= \sum_{n=1}^{\infty} \frac{\epsilon_{t,n+1}(\theta-1)}{\gamma[\kappa-(\theta-1)]-(\theta-1)} \left[\left(\frac{1}{n}\right)^{\frac{\gamma[\kappa-(\theta-1)]-(\theta-1)}{\theta-1}} - \left(\frac{1}{n+1}\right)^{\frac{\gamma[\kappa-(\theta-1)]-(\theta-1)}{\theta-1}} \right].
\end{aligned}
$$

Returning to the expression for σ^2:

$$
\begin{aligned}
\sigma^2 = \mathrm{Var}\left(Z_t^{\theta-1}\right) = &\left(\frac{\kappa}{\kappa-(\theta-1)}\right)^2 \left(\frac{\theta}{\mu_\epsilon}\right)^{-2\frac{\kappa-(\theta-1)}{\theta-1}} \\
&\times \mathrm{Var}\left(\sum_{n=1}^{\infty} \frac{\epsilon_{t,n+1}(\theta-1)}{\gamma[\kappa-(\theta-1)]-(\theta-1)} \left[\left(\frac{1}{n}\right)^{\frac{\gamma[\kappa-(\theta-1)]-(\theta-1)}{\theta-1}} \left(\frac{1}{n+1}\right)^{\frac{\gamma[\kappa-(\theta-1)]-(\theta-1)}{\theta-1}} \right] \right). \\
= \sigma_\epsilon^2 &\left(\frac{\kappa}{\kappa-(\theta-1)}\right)^2 \left(\frac{\theta}{\mu_\epsilon}\right)^{-2\frac{\kappa-(\theta-1)}{\theta-1}} \left(\frac{(\theta-1)}{\gamma[\kappa-(\theta-1)]-(\theta-1)}\right)^2 \\
&\times \sum_{n=1}^{\infty} \left[\left(\frac{1}{n}\right)^{\frac{\gamma[\kappa-(\theta-1)]-(\theta-1)}{\theta-1}} - \left(\frac{1}{n+1}\right)^{\frac{\gamma[\kappa-(\theta-1)]-(\theta-1)}{\theta-1}} \right]^2.
\end{aligned}
$$

Notice that $\left(\frac{\mu_\epsilon}{\theta}\right)^{\frac{1}{\theta-1}}$ appears on the right-hand side. Substituting it for $\underline{z}$, and collecting parameters,

$$
\sigma^2 = \sigma_\epsilon^2 \zeta_{\sigma 1} \underline{z}^{\zeta_{\sigma 2}},
$$

where

$$\zeta_{\sigma 1} := \left(\frac{\kappa}{\kappa-(\theta-1)}\right)^2 \left(\frac{(\theta-1)}{\gamma[\kappa-(\theta-1)]-(\theta-1)}\right)^2 \sum_{n=1}^{\infty}\left[\left(\frac{1}{n}\right)^{\frac{\gamma[\kappa-(\theta-1)]-(\theta-1)}{\theta-1}} - \left(\frac{1}{n+1}\right)^{\frac{\gamma[\kappa-(\theta-1)]-(\theta-1)}{\theta-1}}\right]^2$$

$$\zeta_{\sigma 1} := 2\big[\kappa-(\theta-1)\big].$$

$\square$

Proposition 4.5 *Let technology sets be those that firms choose in the non-stochastic steady state. Then the covariance between firm and aggregate productivity, denoted by* $\sigma_{\omega\Omega}(\omega) = \mathrm{Cov}\Big(Z_t(\omega)^{\theta-1}, Z_t^{\theta-1}\Big)$*, is given by*

$$\sigma_{\omega\Omega}(\omega) = z(\omega)^{\theta-1}\zeta_{\omega\Omega 1}\left[1-\left(\frac{\underline{z}}{z(\omega)}\right)^{\zeta_{\omega\Omega 2}}\right] \tag{61}$$

The covariance between firm and aggregate productivity, expressed as a fraction of firm market value, is approximated to a first order by

$$\frac{\sigma_{\omega\Omega}(\omega)}{V_t(\omega)} \approx \frac{1}{Y_t}\left(\frac{\zeta_{\omega\Omega 1}\left[1-\left(\frac{\underline{z}}{z(\omega)}\right)^{\zeta_{\omega\Omega 2}}\right]}{\zeta_{V1}\left(\frac{z(\omega)}{\underline{z}}\right)^{\zeta_{V2}} + \zeta_{V3}\left(\frac{1}{z(\omega)}\right)^{\zeta_{V4}} - \left(\frac{1}{\underline{z}}\right)^{\zeta_{V4}}}\right). \tag{62}$$

Under parameter restrictions, covariance-over-value falls for all $z(\omega)$ *above a threshold. The ratio also falls in the level of aggregate output.*

Proof. To start, identify a specific firm ω_1, use the definitions of $Z_t(\omega_1)$ and Z_t in the covariance expression, and the cut-offs $\underline{z}$ and $\overline{v}(\omega)$ for the integral bounds:

$$\sigma_{\omega\Omega}(\omega) = \mathrm{Cov}\Big(Z_t(\omega_1)^{\theta-1}, Z_t^{\theta-1}\Big) = \mathrm{Cov}\left(\int\limits_{\mathcal{V}_t(\omega_1)} [z(\omega_1)z_t(v)]^{\theta-1}\lambda(dv), \int\limits_{\Omega} Z_t(\omega)^{\theta-1}\lambda(d\omega)\right)$$

$$= \mathrm{Cov}\left(\int\limits_{\underline{v}=1}^{\overline{v}(\omega_1)} [z(\omega)z_t(v)]^{\theta-1}\lambda(dv), \int\limits_{\underline{z}}^{\infty}\int\limits_{\underline{v}=1}^{\overline{v}(\omega)} [z(\omega)z_t(v)]^{\theta-1}\lambda(dvd\omega)\right).$$

Now consider the first integral:

$$\int\limits_{\underline{v}=1}^{\overline{v}(\omega_1)} [z(\omega_1)z_t(v)]^{\theta-1}\lambda(dv) = z(\omega_1)^{\theta-1}\int\limits_{\underline{v}=1}^{\overline{v}(\omega_1)} \epsilon_{t,\lceil v\rceil}\lambda(dv)$$

$$= z(\omega_1)^{\theta-1}\left[\int\limits_{1}^{2} \epsilon_{t,2}\lambda(dv) + \int\limits_{2}^{3} \epsilon_{t,3}\lambda(dv) + \cdots + \int\limits_{\overline{v}(\omega)-1}^{\overline{v}(\omega)} \epsilon_{t,\overline{v}(\omega)}\lambda(dv)\right]$$

$$= z(\omega_1)^{\theta-1}\sum_{n=1}^{\overline{v}(\omega)-1} \epsilon_{t,n+1},$$

where I have assumed w.l.g. that $\overline{v}(\omega) \in \mathbb{N}$.

Now consider the second integral:

$$\int_{\underline{z}}^{\infty}\int_{\underline{v}=1}^{\overline{v}(\omega)}[z(\omega)z_t(v)]^{\theta-1}\lambda(dvd\omega)=\int_{\underline{v}=1}^{\infty}z_t(v)^{\theta-1}\left(\int_{\underline{z}(v)}^{\infty}z(\omega)^{\theta-1}\lambda(d\omega)\right)\lambda(dv)$$
$$=\int_{\underline{v}}^{\infty}z_t(v)^{\theta-1}\left(\int_{\underline{z}(v)}^{\infty}z(\omega)^{\theta-1}h(z(\omega))\lambda(dz(\omega))\right)\lambda(dv)$$
$$=\int_{\underline{v}}^{\infty}z_t(v)^{\theta-1}\frac{\kappa}{\kappa-(\theta-1)}\underline{z}(v)^{-[\kappa-(\theta-1)]}\lambda(dv),$$

where line two changes measure from Lebesgue to Pareto. Continuing with the second integral, using $\underline{z}(v)=\left(\frac{\theta\mu}{\mu_\epsilon}\right)^{\frac{1}{\theta-1}}v^{\frac{\gamma}{\theta-1}}$,

$$\int_{\underline{z}}^{\infty}\int_{\underline{v}=1}^{\overline{v}(\omega)}[z(\omega)z_t(v)]^{\theta-1}\lambda(dvd\omega)=\left(\frac{\kappa}{\kappa-(\theta-1)}\right)\left(\frac{\theta\mu}{\mu_\epsilon}\right)^{\frac{-[\kappa-(\theta-1])}{\theta-1}}\int_{\underline{v}}^{\infty}z_t(v)^{\theta-1}v^{\frac{-\gamma[\kappa-(\theta-1)]}{\theta-1}}\lambda(dv).$$

Now the single integral on the right-hand side:

$$\int_{\underline{v}}^{\infty}z_t(v)^{\theta-1}v^{\frac{-\gamma[\kappa-(\theta-1)]}{\theta-1}}\lambda(dv)=\int_{\underline{v}=1}^{\infty}\epsilon_{t,\lceil v\rceil}^{\theta-1}v^{\frac{-\gamma[\kappa-(\theta-1)]}{\theta-1}}\lambda(dv)$$
$$=\epsilon_{t,2}\int_{1}^{2}v^{\frac{-\gamma[\kappa-(\theta-1)]}{\theta-1}}\lambda(dv)+\epsilon_{t,3}\int_{2}^{3}v^{\frac{-\gamma[\kappa-(\theta-1)]}{\theta-1}}\lambda(dv)+\dots$$

.

Now consider the integrals of the form:

$$\int_{n}^{n+1} v^{\frac{\gamma[\kappa-(\theta-1)]}{\theta-1}} \lambda(dv) = \left[\left(\frac{\theta-1}{\gamma[\kappa-(\theta-1)]+(\theta-1)} \right) v^{\frac{\gamma[\kappa-(\theta-1)]+(\theta-1)}{\theta-1}} \right]_{n}^{n+1}$$

$$= \left(\frac{\theta-1}{\gamma[\kappa-(\theta-1)]+(\theta-1)} \right) \left[\left(\frac{1}{n} \right)^{\frac{\gamma[\kappa-(\theta-1)]+(\theta-1)}{\theta-1}} - \left(\frac{1}{n+1} \right)^{\frac{\gamma[\kappa-(\theta-1)]+(\theta-1)}{\theta-1}} \right].$$

So the single integral becomes:

$$\int_{\underline{v}}^{\infty} z_t(v)^{\theta-1} v^{\frac{-\gamma[\kappa-(\theta-1)]}{\theta-1}} \lambda(dv) = \left(\frac{\theta-1}{\gamma[\kappa-(\theta-1)]+(\theta-1)} \right)$$

$$\times \sum_{n=1}^{\infty} \epsilon_{t,n+1} \left[\left(\frac{1}{n} \right)^{\frac{\gamma[\kappa-(\theta-1)]+(\theta-1)}{\theta-1}} - \left(\frac{1}{n+1} \right)^{\frac{\gamma[\kappa-(\theta-1)]+(\theta-1)}{\theta-1}} \right],$$

and the second integral becomes:

$$\int_{\underline{z}}^{\infty} \int_{\underline{v}=1}^{\overline{v}(\omega)} [z(\omega) z_t(v)]^{\theta-1} \lambda(dv d\omega) = \left(\frac{\kappa}{\kappa-(\theta-1)} \right) \left(\frac{\theta\mu}{\mu_\epsilon} \right)^{\frac{-[\kappa-(\theta-1])}{\theta-1}} \left(\frac{\theta-1}{\gamma[\kappa-(\theta-1)]+(\theta-1)} \right)$$

$$\times \sum_{n=1}^{\infty} \epsilon_{t,n+1} \left[\left(\frac{1}{n} \right)^{\frac{\gamma[\kappa-(\theta-1)]+(\theta-1)}{\theta-1}} - \left(\frac{1}{n+1} \right)^{\frac{\gamma[\kappa-(\theta-1)]+(\theta-1)}{\theta-1}} \right].$$

Now, recall that $\text{Cov}\big(\epsilon_{t,n}, \epsilon_{t,m}\big) = 0 \; \forall \; n \neq m$, and write the desired covariance as:

$$\begin{aligned}
\sigma_{\omega\Omega}(\omega) &= \text{Cov}\Big(Z_t(\omega_1)^{\theta-1}, Z_t^{\theta-1}\Big) \\
&= z(\omega_1)^{\theta-1}\left(\frac{\kappa}{\kappa-(\theta-1)}\right)\left(\frac{\theta\mu}{\mu_\epsilon}\right)^{\frac{-[\kappa-(\theta-1])}{\theta-1}}\left(\frac{\theta-1}{\gamma[\kappa-(\theta-1)]+(\theta-1)}\right) \\
&\times \text{Cov}\left(\sum_{n=1}^{\overline{v}(\omega)-1}\epsilon_{t,n+1}, \sum_{n=1}^{\infty}\epsilon_{t,n+1}\left[\left(\frac{1}{n}\right)^{\frac{\gamma[\kappa-(\theta-1)]+(\theta-1)}{\theta-1}} - \left(\frac{1}{n+1}\right)^{\frac{\gamma[\kappa-(\theta-1)]+(\theta-1)}{\theta-1}}\right]\right)
\end{aligned}$$

$$\begin{aligned}
&= z(\omega_1)^{\theta-1}\left(\frac{\kappa}{\kappa-(\theta-1)}\right)\left(\frac{\theta\mu}{\mu_\epsilon}\right)^{\frac{-[\kappa-(\theta-1])}{\theta-1}}\left(\frac{\theta-1}{\gamma[\kappa-(\theta-1)]+(\theta-1)}\right) \\
&\qquad \times \sum_{n=1}^{\overline{v}(\omega)-1}\left[\left(\frac{1}{n}\right)^{\frac{\gamma[\kappa-(\theta-1)]+(\theta-1)}{\theta-1}} - \left(\frac{1}{n+1}\right)^{\frac{\gamma[\kappa-(\theta-1)]+(\theta-1)}{\theta-1}}\right] \\
&\qquad \times \text{Cov}\big(\epsilon_{t,n+1}, \epsilon_{t,n+1}\big)
\end{aligned}$$

$$\begin{aligned}
&= \sigma_\epsilon^2 z(\omega_1)^{\theta-1}\left(\frac{\kappa}{\kappa-(\theta-1)}\right)\left(\frac{\theta\mu}{\mu_\epsilon}\right)^{\frac{-[\kappa-(\theta-1])}{\theta-1}}\left(\frac{\theta-1}{\gamma[\kappa-(\theta-1)]+(\theta-1)}\right) \\
&\qquad \times \sum_{n=1}^{\overline{v}(\omega)-1}\left[\left(\frac{1}{n}\right)^{\frac{\gamma[\kappa-(\theta-1)]+(\theta-1)}{\theta-1}} - \left(\frac{1}{n+1}\right)^{\frac{\gamma[\kappa-(\theta-1)]+(\theta-1)}{\theta-1}}\right].
\end{aligned}$$

Notice that the right-hand side summation, with a as a temporary placeholder, is of form:

$$\begin{aligned}
\sum_{n=1}^{\overline{v}(\omega_1)-1}\left[\left(\frac{1}{n}\right)^a - \left(\frac{1}{n+1}\right)^a\right] &= \left[\left(\frac{1}{1}\right)^a - \left(\frac{1}{2}\right)^a + \left(\frac{1}{2}\right)^a - \left(\frac{1}{3}\right)^a + \cdots - \left(\frac{1}{\overline{v}(\omega_1)}\right)^a\right] \\
&= \left[1 - \left(\frac{1}{\overline{v}(\omega_1)}\right)^a\right].
\end{aligned}$$

Returning to the covariance expression, and simplifying the summation as above,

$$
\begin{aligned}
\sigma_{\omega\Omega}(\omega) &= \mathrm{Cov}\Big(Z_t(\omega_1)^{\theta-1}, Z_t^{\theta-1}\Big) \\
&= \sigma_\epsilon^2 z(\omega_1)^{\theta-1}\left(\frac{\kappa}{\kappa-(\theta-1)}\right)\left(\frac{\theta\mu}{\mu_\epsilon}\right)^{\frac{-[\kappa-(\theta-1])}{\theta-1}}\left(\frac{\theta-1}{\gamma[\kappa-(\theta-1)]+(\theta-1)}\right) \\
&\qquad \times\left[1-\left(\frac{1}{\overline{v}(\omega_1)}\right)^{\frac{\gamma[\kappa-(\theta-1)]+(\theta-1)}{\theta-1}}\right] \\
&= \sigma_\epsilon^2 z(\omega_1)^{\theta-1}\left(\frac{\kappa}{\kappa-(\theta-1)}\right)\left(\frac{\theta\mu}{\mu_\epsilon}\right)^{\frac{-[\kappa-(\theta-1)]}{\theta-1}}\left(\frac{\theta-1}{\gamma[\kappa-(\theta-1)]+(\theta-1)}\right) \\
&\qquad \times\left[1-\left(\frac{1}{\overline{v}(\omega_1)}\right)^{\frac{\gamma[\kappa-(\theta-1)]+(\theta-1)}{\theta-1}}\right] \\
&= \sigma_\epsilon^2 z(\omega_1)^{\theta-1}\left(\frac{\kappa}{\kappa-(\theta-1)}\right)\left(\frac{\theta\mu}{\mu_\epsilon}\right)^{\frac{-[\kappa-(\theta-1])}{\theta-1}}\left(\frac{\theta-1}{\gamma[\kappa-(\theta-1)]+(\theta-1)}\right) \\
&\qquad \times\left[1-\left(\frac{\theta\mu}{\mu_\epsilon}\right)^{\frac{\gamma[\kappa-(\theta-1)]}{\gamma(\theta-1)}}\left(\frac{1}{z(\omega_1)}\right)^{\frac{\gamma[\kappa-(\theta-1)]}{\gamma}}\right]
\end{aligned}
$$

where the last line uses $\overline{v}(\omega_1) = \left(\frac{\mu_\epsilon}{\theta\mu}\right)^{\frac{1}{\gamma}} z(\omega_1)^{\frac{\theta-1}{\gamma}}$.

Finally, collect parameters, return from specific ω_1 to arbitrary ω, and write:

$$\frac{\sigma_{\omega\Omega}(\omega)}{\sigma_\epsilon^2} = z(\omega)^{\theta-1}\zeta_{\omega\Omega1}\left[1 - \left(\frac{\underline{z}}{z(\omega)}\right)^{\zeta_{\omega\Omega2}}\right], \text{ where}$$

$$\zeta_{\omega\Omega1} = \left(\frac{\kappa}{\kappa-(\theta-1)}\right)\left(\frac{\theta\mu}{\mu_\epsilon}\right)^{\frac{-[\kappa-(\theta-1)]}{\theta-1}}\left(\frac{\theta-1}{\gamma[\kappa-(\theta-1)]+(\theta-1)}\right)$$

$$\zeta_{\omega\Omega1} = \frac{\gamma[\kappa-(\theta-1)]}{\gamma}.$$

Recall that the μ appearing in $\zeta_{\omega\Omega1}$ has already been expressed in terms of parameters, so the above expression suffices.

Now turn to covariance over market value. Start from the following primitives:

$$V_t(\omega) = \mathrm{E}_t\left[\sum_{s=t+1}^{\infty}\beta^{s-t}\frac{u'(C_s)}{u'(C_t)}\big(\Pi_s(\omega) - F_s(\omega)\big)\right]$$

$$\Pi_s(\omega) = \int_{\mathcal{V}(\omega)} \pi_s(v,\omega)\lambda(dv) = \frac{1}{\theta}\left(\frac{Z_s(\omega)}{Z_s}\right)^{\theta-1} Y_s$$

$$F_s(\omega) = \int_{\mathcal{V}(\omega)} \frac{Y_s}{\mu}v^\gamma\lambda(dv).$$

Using $u(C_s) = \ln(C_s)$ and above primitives, rearrange to get:

$$\frac{V_t(\omega)}{Y_t} = \mathrm{E}\left[\sum_{s=t+1}^{\infty}\beta^{s-t}\left\{\frac{1}{\theta}\left(\frac{z(\omega)}{Z_s}\right)^{\theta-1}\int_{\mathcal{V}(\omega)} z_s(v)^{\theta-1} - \frac{v^\gamma}{\mu}\lambda(dv)\right\}\right].$$

Split up the integral and evaluate the first term, assuming w.l.g. that $\overline{v}(\omega) \in \mathbb{N}$:

$$\begin{aligned}
\int_{\mathcal{V}(\omega)} z_s(v)^{\theta-1}\lambda(dv) &= \int_{\underline{v}=1}^{\overline{v}(\omega)} \epsilon_{s,\lceil v \rceil}\lambda(dv) \\
&= \int_1^2 \epsilon_{s,2}\lambda(dv) + \int_2^3 \epsilon_{s,3}\lambda(dv) + \cdots + \int_{\overline{v}(\omega)-1}^{\overline{v}(\omega)} \epsilon_{s,\overline{v}(\omega)}\lambda(dv) \\
&= \sum_{n=1}^{\overline{v}(\omega)-1} \epsilon_{s,n+1}
\end{aligned}$$

Now evaluate the second part of the integral that we split above:

$$\int_{\mathcal{V}(\omega)} \frac{v^\gamma}{\mu}\lambda(dv) = \frac{1}{\mu}\left(\frac{\overline{v}(\omega)^{\gamma+1}}{\gamma+1} - \frac{1}{\gamma+1}\right).$$

Substituting back into the expression for firm value,

$$\frac{V_t(\omega)}{Y_t} = \sum_{s=t+1}^{\infty} \beta^{s-t}\frac{z(\omega)^{\theta-1}}{\theta}\sum_{n=1}^{\overline{v}(\omega)-1} \mathrm{E}\left[\frac{\epsilon_{s,n+1}}{Z_s^{\theta-1}}\right] - \sum_{s=t+1}^{\infty}\beta^{s-t}\left(\frac{\overline{v}(\omega)^{\gamma+1}}{1+\gamma} - \frac{1}{1+\gamma}\right)$$

To a first-order approximation, the expectation is: $\mathrm{E}\left[\frac{\epsilon_{s,n+1}}{Z_s^{\theta-1}}\right] \approx \frac{\mu_\epsilon}{\mu}$. Simplifying,

$$\frac{V_t(\omega)}{Y_t} \approx z(\omega)^{\frac{1+\gamma}{\gamma}(\theta-1)}\left(\frac{\mu_\epsilon}{\theta\mu}\right)^{\frac{1+\gamma}{\gamma}}\left(\frac{\gamma}{1+\gamma}\right) - z(\omega)^{\theta-1}\left(\frac{\mu_\epsilon}{\theta\mu}\right) + \left(\frac{1}{1+\gamma}\right)$$

Now combining with the covariance expression derived above:

$$\frac{\sigma_{\omega\Omega}(\omega)}{V_t(\omega)} \approx \frac{\sigma_\epsilon^2}{Y_t}\cdot\frac{z(\omega)^{\theta-1}\left(\frac{\theta-1}{\gamma[\kappa-(\theta-1)]-(\theta-1)}\right)\left[1-\left(\frac{\theta\mu}{\mu_\epsilon}\right)^{\frac{\gamma[\kappa-(\theta-1)]-(\theta-1)}{\gamma(\theta-1)}}\left(\frac{1}{z(\omega)}\right)^{\frac{\gamma[\kappa-(\theta-1)]-(\theta-1)}{\gamma}}\right]}{z(\omega)^{\frac{1+\gamma}{\gamma}(\theta-1)}\left(\frac{\mu_\epsilon}{\theta\mu}\right)^{\frac{1+\gamma}{\gamma}}\left(\frac{\gamma}{1+\gamma}\right) - z(\omega)^{\theta-1}\left(\frac{\mu_\epsilon}{\theta\mu}\right) + \frac{1}{1+\gamma}}.$$

Finally, using the expression for $\underline{z}$, and collecting parameters to simplify,

$$\frac{\sigma_{\omega\Omega}(\omega)}{V_t(\omega)} \approx \frac{1}{Y_t}\left(\frac{\zeta_{\omega\Omega1}\left[1-\left(\frac{\underline{z}}{z(\omega)}\right)^{\zeta_{\omega\Omega2}}\right]}{\zeta_{V1}\left(\frac{z(\omega)}{\underline{z}}\right)^{\zeta_{V2}}+\zeta_{V3}\left(\frac{1}{z(\omega)}\right)^{\zeta_{V4}}-\left(\frac{1}{\underline{z}}\right)^{\zeta_{V4}}}\right), \text{ where}$$

$$\zeta_{\omega\Omega1} := \underline{z}^{\theta-1}\left(\frac{\sigma_\epsilon^2(\theta-1)}{\gamma[\kappa-(\theta-1)]-(\theta-1)}\right), \qquad \zeta_{\omega\Omega2} := \left(\frac{\gamma[\kappa-(\theta-1)]-(\theta-1)}{\gamma}\right)$$

$$\zeta_{V1} = \left(\frac{\gamma}{1+\gamma}\right), \qquad \zeta_{V2} := \left(\frac{\theta-1}{\gamma}\right), \qquad \zeta_{V3} := \left(\frac{1}{\gamma+1}\right), \qquad \zeta_{V4} := (\theta-1).$$

□

Proposition 4.6 *Let technology sets be those that firms choose in the non-stochastic steady state, and let $Z_t(\omega_1)$ and $Z_t(\omega_2)$ be firm productivities for firms ω_1 and ω_2, where $Z_t(\omega_1) < Z_t(\omega_2)$. Then the correlation between firm productivities is given by*

$$\mathrm{Corr}\big(Z_t(\omega_1), Z_t(\omega_2)\big) = blah, \tag{63}$$

and the correlation $\mathrm{Corr}\big(Z_t(\omega_1), Z_t(\omega_2)\big)$ is decreasing in the distance between productivities, $|z(\omega_1) - z(\omega_2)|$.

Proof. □

Proposition 4.7 *Let technology sets be those that firms choose in the non-stochastic steady state. Then firm ω's expected excess return is approximated to a second order by*

$$\mathrm{E}_t\big[r_{t+1}(\omega) - r_{f,t+1}\big] \approx \zeta_{r1}\frac{\mu(\omega)}{V_t(\omega)} + \zeta_{r2}\frac{\sigma_{\omega\Omega}(\omega)}{V_t(\omega)}, \tag{64}$$

where I define firm ω's return as $r_t(\omega) = \left[V_{t+1}(\omega) + \Pi_{t+1}(\omega) - F_{t+1}(\omega)\right]/V_t(\omega)$, and the risk-free rate as $r_{f,t} = m_{t,t+1}^{-1}$. Under parameter restrictions, expected excess returns decrease in firm productivity $z(\omega)$ for all $z(\omega)$ above a threshold.

Proof. Start with the definition of firm ω's stock return:

$$
\begin{aligned}
r_{t+1}(\omega) &= \frac{V_{t+1}(\omega) + \Pi_{t+1}(\omega) - F_{t+1}(\omega)}{V_t(\omega)} \\
&= \frac{\mathrm{E}\left[\sum_{s=t+2}^{\infty} m_{t+1,s}(\Pi_s(\omega) - F_s(\omega))\right] + \Pi_{t+1}(\omega) - F_{t+1}(\omega)}{V_t(\omega)} \\
&= \frac{Y_{t+1}\mathrm{E}\left[\sum_{s=t+1}^{\infty} \beta^{s-(t+1)} \left(\frac{\Pi_s(\omega)}{Y_s} - \frac{F_s(\omega)}{Y_s}\right)\right]}{V_t(\omega)},
\end{aligned}
$$

where the third line assumes log utility and uses the definition of the household stochastic discount factor. Now take the time-t conditional expectation:

$$\begin{aligned}
\mathrm{E}_t\big[r_{t+1}(\omega)\big] &= \mathrm{E}_t\left[\frac{Y_{t+1}\mathrm{E}_{t+1}\left[\sum_{s=t+1}^{\infty}\beta^{s-(t+1)}\left(\frac{\Pi_s(\omega)}{Y_s}-\frac{F_s(\omega)}{Y_s}\right)\right]}{V_t(\omega)}\right] \\
&= \frac{\mathrm{E}_t[Y_{t+1}]\mathrm{E}_t\left[\mathrm{E}_{t+1}\left[\sum_{s=t+1}^{\infty}\beta^{s-(t+1)}\left(\frac{\Pi_s(\omega)}{Y_s}-\frac{F_s(\omega)}{Y_s}\right)\right]\right]}{V_t(\omega)} \\
&\qquad + \frac{\mathrm{Cov}_t\left(Y_{t+1},\mathrm{E}_{t+1}\left[\sum_{s=t+1}^{\infty}\beta^{s-(t+1)}\left(\frac{\Pi_s(\omega)}{Y_s}-\frac{F_s(\omega)}{Y_s}\right)\right]\right)}{V_t(\omega)} \\
&= \frac{\mathrm{E}_t[Y_{t+1}]}{\beta Y_t} + \frac{\mathrm{Cov}_t\left(Y_{t+1},\mathrm{E}_{t+1}\left[\sum_{s=t+1}^{\infty}\beta^{s-(t+1)}\left(\frac{\Pi_s(\omega)}{Y_s}-\frac{F_s(\omega)}{Y_s}\right)\right]\right)}{V_t(\omega)} \\
\Leftrightarrow \quad \mathrm{E}_t\big[r_{t+1}(\omega)-r_{f,t+1}\big] &= \frac{\mathrm{Cov}_t\left(Y_{t+1},\mathrm{E}_{t+1}\left[\sum_{s=t+1}^{\infty}\beta^{s-(t+1)}\left(\frac{\Pi_s(\omega)}{Y_s}-\frac{F_s(\omega)}{Y_s}\right)\right]\right)}{V_t(\omega)}.
\end{aligned}$$

Consider the covariance term separately, and recall that zero serial correlation has been assumed for the random $\epsilon_{s,n}$'s:

$$\begin{aligned}
&\mathrm{Cov}_t\left(Y_{t+1}, \mathrm{E}_{t+1}\left[\sum_{s=t+1}^{\infty}\beta^{s-(t+1)}\left(\frac{\Pi_s(\omega)}{Y_s}-\frac{F_s(\omega)}{Y_s}\right)\right]\right)\\
&\qquad=\mathrm{Cov}_t\left(Y_{t+1}, \left(\frac{\Pi_{t+1}(\omega)}{Y_{t+1}}-\frac{F_{t+1}(\omega)}{Y_{t+1}}\right)\right)\\
&\qquad=\mathrm{E}_t\left[\Pi_{t+1}(\omega)-F_{t+1}(\omega)\right]-\mathrm{E}_t[Y_{t+1}]\mathrm{E}_t\left[\frac{\Pi_{t+1}(\omega)}{Y_{t+1}}-\frac{F_{t+1}(\omega)}{Y_{t+1}}\right]\\
&\qquad=\mathrm{E}_t\left[\Pi_{t+1}(\omega)\right]-\mathrm{E}_t[Y_{t+1}]\mathrm{E}_t\left[\frac{\Pi_{t+1}(\omega)}{Y_{t+1}}\right]\\
&\qquad=\mathrm{E}_t\left[Y_{t+1}\int_{\mathcal{V}(\omega)}\frac{1}{\theta}\left(\frac{z(\omega)z_{t+1}(v)}{Z_{t+1}}\right)^{\theta-1}\lambda(dv)\right]-\mathrm{E}_t[Y_{t+1}]\mathrm{E}_t\left[\int_{\mathcal{V}(\omega)}\frac{1}{\theta}\left(\frac{z(\omega)z_{t+1}(v)}{Z_{t+1}}\right)^{\theta-1}\lambda(dv)\right]\\
&\qquad=\mathrm{E}_t\left[\frac{1}{\theta}Y_{t+1}\left(\frac{Z_{t+1}(\omega)}{Z_{t+1}}\right)^{\theta-1}\right]-\mathrm{E}_t[Y_{t+1}]\mathrm{E}_t\left[\frac{1}{\theta}\left(\frac{Z_{t+1}(\omega)}{Z_{t+1}}\right)^{\theta-1}\right],
\end{aligned}$$

where the second line uses the assumption of zero serial correlation. Now recall from (53) that $Y_{t+1}=Z_{t+1}(K_{t+1})^{\alpha}(L)^{1-\alpha}$, that K_{t+1} is determined in t, and that L fixed, so

$$\begin{aligned}
&\mathrm{Cov}_t\left(Y_{t+1}, \mathrm{E}_{t+1}\left[\sum_{s=t+1}^{\infty}\beta^{s-(t+1)}\left(\frac{\Pi_s(\omega)}{Y_s}-\frac{F_s(\omega)}{Y_s}\right)\right]\right)\\
&\qquad=\frac{(K_{t+1}^{\alpha}L^{1-\alpha})}{\theta}\left(\mathrm{E}_t\left[\frac{Z_{t+1}(\omega)^{\theta-1}}{Z_{t+1}^{\theta-2}}\right]-\mathrm{E}_t[Z_{t+1}]\mathrm{E}_t\left[\left(\frac{Z_{t+1}(\omega)}{Z_{t+1}}\right)^{\theta-1}\right]\right)
\end{aligned}$$

Next, second-order approximate the individual right-hand side expectations around the non-stochastic steady state values $\mu(\omega)$ and μ. Starting with the first right-hand side expectation:

$$\mathrm{E}_t\left[\frac{Z_{t+1}(\omega)^{\theta-1}}{Z_{t+1}^{\theta-2}}\right]\approx\frac{\mu(\omega)}{\mu^{\frac{\theta-2}{\theta-1}}}+\frac{1}{2}\left(\frac{\theta-2}{\theta-1}\right)\left(\frac{\theta-2}{\theta-1}+1\right)\frac{\mu(\omega)}{\mu^{\frac{\theta-2}{\theta-1}+2}}\cdot\sigma^2-\frac{1}{2}\left(\frac{\theta-2}{\theta-1}\right)\frac{1}{\mu^{\frac{\theta-2}{\theta-1}+1}}\cdot\sigma_{\omega\Omega}(\omega).$$

Now the second:

$$\mathrm{E}_t[Z_{t+1}] \approx \mu^{\frac{1}{\theta-1}} + \frac{1}{2}\left(\frac{1}{\theta-1}\right)\left(\frac{1}{\theta-1}-1\right)\mu^{\frac{1}{\theta-1}-2}\sigma^2.$$

And the third:

$$\mathrm{E}_t\left[\left(\frac{Z_t(\omega)}{Z_t}\right)^{\theta-1}\right] \approx \left(\frac{1}{\mu}+\frac{\sigma^2}{\mu^3}\right)\mu(\omega) - \left(\frac{1}{2\mu^2}\right)\sigma_{\omega\Omega}(\omega).$$

Substituting the approximations in the covariance expression, and rearranging,

$$\begin{aligned}
&\mathrm{Cov}_t\left(Y_{t+1}, \mathrm{E}_{t+1}\left[\sum_{s=t+1}^{\infty}\beta^{s-(t+1)}\left(\frac{\Pi_s(\omega)}{Y_s}-\frac{F_s(\omega)}{Y_s}\right)\right]\right)\\
&\approx \frac{(K_{t+1}^{\alpha}L^{1-\alpha})}{\theta}\left(\left(\frac{\theta-2}{\theta-1}\right)\left(\frac{\mu^{\frac{1}{\theta-1}}}{\mu}\right)\sigma^2 + \left(\frac{\mu^{\frac{1}{\theta-1}}}{\mu}\right)\left(\frac{\sigma}{\mu}\right)^2\left[\left(\frac{1}{2}\right)\left(\frac{\theta-2}{\theta-1}\right)\left(\frac{1}{\theta-1}\right)\left(\frac{\sigma}{\mu}\right)^2-1\right]\right)\cdot\mu(\omega)\\
&\qquad + \frac{(K_{t+1}^{\alpha}L^{1-\alpha})}{\theta}\left[\left(\frac{1}{2}\right)\left(\frac{1}{\theta-1}\right)\left(\frac{1}{\mu^{\frac{1}{\theta-1}}}\right)\left\{1-\left(\frac{1}{2}\right)\left(\frac{\theta-2}{\theta-1}\right)\left(\frac{1}{\mu^2}\right)\right\}\right]\cdot\sigma_{\omega\Omega}(\omega)\\
&\approx \zeta_{r1}\mu(\omega)+\zeta_{r2}\sigma_{\omega\Omega}(\omega),
\end{aligned}$$

where ζ_{r1} and ζ_{r2} are parameter collections given by:

$$\begin{aligned}
\zeta_{r1} &:= \frac{(K_{t+1}^{\alpha}L^{1-\alpha})}{\theta}\left(\left(\frac{\theta-2}{\theta-1}\right)\left(\frac{\mu^{\frac{1}{\theta-1}}}{\mu}\right)\sigma^2 + \left(\frac{\mu^{\frac{1}{\theta-1}}}{\mu}\right)\left(\frac{\sigma}{\mu}\right)^2\left[\left(\frac{1}{2}\right)\left(\frac{\theta-2}{\theta-1}\right)\left(\frac{1}{\theta-1}\right)\left(\frac{\sigma}{\mu}\right)^2-1\right]\right)\\
\zeta_{r2} &:= \frac{(K_{t+1}^{\alpha}L^{1-\alpha})}{\theta}\left[\left(\frac{1}{2}\right)\left(\frac{1}{\theta-1}\right)\left(\frac{1}{\mu^{\frac{1}{\theta-1}}}\right)\left\{1-\left(\frac{1}{2}\right)\left(\frac{\theta-2}{\theta-1}\right)\left(\frac{1}{\mu^2}\right)\right\}\right],
\end{aligned}$$

and K_{t+1} is evaluated at its steady-state value. Finally, returning to the expression for expected excess returns,

$$\mathrm{E}_t\left[r_{t+1}(\omega) - r_{f,t+1}\right] \approx \zeta_{r1}\frac{\mu(\omega)}{V_t(\omega)} + \zeta_{r2}\frac{\sigma_{\omega\Omega}(\omega)}{V_t(\omega)}.$$

□

A.1.3 Steady-state equilibrium

Equilibrium requires that the following market clearing conditions hold:

$$\begin{aligned}
c_t(v,\omega) &= y_t(v,\omega), \\
L &= \int_{\Omega}\int_{\mathcal{V}(\omega)} l_t(v,\omega)\lambda(dvd\omega) + l_t, \\
K_t &= \int_{\Omega}\int_{\mathcal{V}(\omega)} k_t(v,\omega)\lambda(dvd\omega) + k_t, \\
\tilde{I}_t &= I_t + \int_{\Omega}\int_{\mathcal{V}(\omega)} f_s(v)\lambda(dvd\omega), \\
S_t(\omega) &= 1.
\end{aligned}$$

In the steady state equilibrium, random productivities take their expected values ($(z_t(v)^{\theta-1} = \mu_\epsilon,\ \forall v \in \mathcal{V}$), and capital and consumption are constant over time ($C_{t+1} = C_t = C^*, K_{t+1} = K_t = K^*$). Under these conditions, solving for steady-state values of endogenous variables is straight forward.

Begin by solving for the steady state wage and rental rate. In steady state, (33) becomes

$1 = \beta(1 + r^* - \delta)$. Using (43) to substitute for r^*:

$$\begin{aligned} 1 &= \beta(1 + r^* - \delta) \\ &= \beta(1 - \delta + \alpha\mu\left(\frac{l^*}{k^*}\right)^{1-\alpha}) \\ \Leftrightarrow \qquad \frac{k^*}{l^*} &= \left[\frac{\alpha\beta\mu}{1 - \beta(1-\delta)}\right]^{\frac{1}{1-\alpha}}. \end{aligned}$$

Returning to (43) and evaluating at steady state,

$$\begin{aligned} r^* &= \alpha\mu\left(\frac{k^*}{l^*}\right)^{\alpha-1} \\ &= \mu\left[\frac{\alpha\beta\mu}{1 - \beta(1-\delta)}\right]^{\frac{\alpha-1}{1-\alpha}} \\ &= \frac{1 - \beta(1-\delta)}{\beta}. \end{aligned}$$

Now using (44),

$$\begin{aligned} w^* &= (1-\alpha)\mu\left(\frac{k^*}{l^*}\right)^{\alpha} \\ &= (1-\alpha)\mu\left[\frac{\alpha\beta\mu}{1 - \beta(1-\delta)}\right]^{\frac{\alpha}{1-\alpha}}. \end{aligned}$$

Combining,

$$\frac{r^*}{w^*} = \left(\frac{\alpha}{1-\alpha}\right)\left[\frac{1 - \beta(1-\delta)}{\alpha\beta\mu}\right]^{\frac{1}{1-\alpha}}.$$

Next, find an expression for the steady-state aggregate capital stock. Start with the definitions of aggregate capital and labor:

$$K^* = \int\limits_{\Omega} \int\limits_{\mathcal{V}(\omega)} k^*(v,\omega)\lambda(dvd\omega) + k^*,$$
$$L = \int\limits_{\Omega} \int\limits_{\mathcal{V}(\omega)} l^*(v,\omega)\lambda(dvd\omega) + l^*.$$

Now use (42) and (45) to write

$$\begin{aligned} L &= \left(\frac{1-\alpha}{\alpha}\right)\left(\frac{w^*}{r^*}\right)\left[\int\limits_{\Omega} \int\limits_{\mathcal{V}(\omega)} k^*(v,\omega)\lambda(dvd\omega) + k^*\right] \\ \Leftrightarrow \qquad K^* &= \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{r^*}{w^*}\right)L \\ &= \left(\frac{\alpha}{1-\alpha}\right)^2\left[\frac{1-\beta(1-\delta)}{\alpha\beta\mu}\right]^{\frac{1}{1-\alpha}}L. \end{aligned}$$

Recall that L is exogenous, so the above expression suffices. Next, use the law of motion for capital to find a steady-state expression for investment demand I_t:

$$\begin{aligned} K^* &= I^* + (1-\delta)K^* \\ \Leftrightarrow I^* &= \delta K^* \\ &= \delta\left(\frac{\alpha}{1-\alpha}\right)^2\left[\frac{1-\beta(1-\delta)}{\alpha\beta\mu}\right]^{\frac{1}{1-\alpha}}L. \end{aligned}$$

A.2 Discussion of the Data

A.2.1 Productivity estimation

I follow the procedures in Olley and Pakes (1996) and İmrohoroğlu and Tuzel (2014), and estimate a Cobb-Douglas production function in log form. The estimation equation is given by:

$$\ln(Y_{\omega,t}) = \alpha_0 + \alpha_K \ln(K_{\omega,t}) + \alpha_L \ln(L_{\omega,t}) + \ln(Z_{\omega,t}), \tag{65}$$

where the residual $Z_{\omega,t}$ is firm-level total factor productivity. The procedure assumes $Z_{\omega,t} = \xi_{\omega,t}\eta_{\omega,t}$, where $Z_{\omega,t}$ is unknown to the econometrician, but $\xi_{\omega,t}$ is known to the firm.

Olley and Pakes (1996) use a simple behavioral model to derive reduced-form decision rules for firms deciding each period whether to exit or continue producing, and if continuing, how much new capital to purchase. Firms' decision rules depend on their current knowledge of productivity $\xi_{\omega,t}$. Each firm's exit decision is captured by an indicator function $\chi_{\omega,t}$:

$$\chi_{\omega,t} = \begin{cases} 1 & \text{if } \xi_{\omega,t} > \underline{\xi}_{\omega,t} \\ 0 & \text{otherwise,} \end{cases} \tag{66}$$

and their investment decision is captured by an investment function:

$$\ln(I_{\omega,t}) = \ln(I_{\omega,t})\big(\ln(\xi_{\omega,t}), \ln(K_{\omega,t})\big). \tag{67}$$

Inverting the investment function, $\ln(\xi_{\omega,t}) = \ln(\xi_{\omega,t})\big(\ln(K_{\omega,t}), \ln(I_{\omega,t})\big)$. Defining a new function, $\phi\big(\ln(K_{\omega,t}), \ln(I_{\omega,t})\big) = \alpha_0 + \alpha_K \ln(K_{\omega,t}) + \ln(\xi_{\omega,t})\big(\ln(K_{\omega,t}), \ln(I_{\omega,t})\big)$, the productivity regression equation (65) becomes

$$\ln(Y_{\omega,t}) = \alpha_L \ln(L_{\omega,t}) + \phi\big(\ln(K_{\omega,t}), \ln(I_{\omega,t})\big) + \ln(\eta_{\omega,t}), \tag{68}$$

In a first stage regression, equation (68) is estimated by least squares, where the function $\phi\big(\ln(K_{\omega,t}), \ln(I_{\omega,t})\big)$ controls for the forecastable component of firm productivity, and is approximated by a polynomial in $\ln(K_{\omega,t})$ and $\ln(I_{\omega,t})$, denoting a firm's capital stock and investment. I include time-industry controls in this stage to prevent time-industry effects from influencing the first-stage estimates. The remaining estimation equations are given by:

$$P_{i,t} = \mathscr{P}_t(i_{i,t}, k_{i,t}) \tag{69}$$

$$\ln(Y_{\omega,t}) - \alpha_L \ln(L_{\omega,t}) = \alpha_L \ln(L_{\omega,t}) + g(P_{i,t}, \phi_{i,t} - \beta_k k_{i,t}) + \ln(\xi_{\omega,t+1}) + \ln(\eta_{\omega,t+1}). \tag{70}$$

In the second stage, each firm's probability of exit is estimated by equation (69) using probit, where $\mathscr{P}_t(i_{i,t}, k_{i,t})$ is approximated by a polynomial in i_t and k_t. Finally, equation (70) is estimated by non-linear least squares, using estimates from stages one and two for $P_{i,t}$ and $\phi_{i,t}$, and approximating $g(P_{i,t}, \phi_{i,t} - \beta_k k_{i,t})$ non-parametrically. The non-parametric functions ϕ, $\mathscr{P}$, and g are derived in greater detail in Olley and Pakes (1996).

I map model variables to Compustat variables in the following way, writing Compustat variables in fixed-width font: labor expense is $L_{\omega,t} = \texttt{WAGE} \times \texttt{EMP}$; capital is $K_{\omega,t} = \texttt{L.PPENT}$, value added is $Y_{\omega,t} = \texttt{OIBDP} + \texttt{WAGE} \times \texttt{EMP}$.

İmrohoroğlu and Tuzel (2014) use an expanding estimation window to prevent information that would have been unavailable to market participants in a particular period from distorting results when they combine estimated productivity with financial market data. I find that the expanding window approach leads to large differences in the volatility of production function estimates in earlier periods relative to later periods. This increased volatility biases the rolling-window covariance estimates in early years, so I instead use the full sample period to estimate production function parameters, and then compute productivity as the residual each period.

References

Acemoglu, D., Carvalho, V., & Ozdaglar, A. (2012). The network origins of aggregate fluctuations. *Econometrica*, *80*(5), 1977–2016.

Ansoff, H. I. (1957). Strategies for diversification. *Harvard business review*, *35*(5), 113–124.

Baily, M. N., Hulten, C., & Campbell, D. (1992). Productivity dynamics in manufacturing plants. *Brookings Papers on Economic Activity. Microeconomics*, *1992*, 187.

Baqaee, D. R. & Farhi, E. (2019). *The macroeconomic impact of microeconomic shocks: Beyond hulten's theorem*. National Bureau of Economic Research.

Bartelsman, E. & Doms, M. (2000). Understanding productivity: Lessons from longitudinal microdata. *Journal of Economic literature*, *38*(3), 569–594.

Bartelsman, E., Haltiwanger, J., & Scarpetta, S. (2009). *Cross-country differences in productivity: The role of allocation and selection*. National Bureau of Economic Research.

Berk, J. B., Green, R. C., & Naik, V. (1999). Optimal investment, growth options, and security returns. *The Journal of Finance*, *54*(5), 1553–1607.

Bernard, A. B., Redding, S. J., & Schott, P. K. (2010). Multiple-product firms and product switching. *American Economic Review*, *100*(1), 70–97.

Bilbiie, F. O., Ghironi, F., & Melitz, M. J. (2012). Endogenous entry, product variety, and business cycles. *Journal of Political Economy*, *120*(2), 304–345.

Bloom, N., Sadun, R., & Reenen, J. V. (2016). *Management as a technology?* National Bureau of Economic Research.

Boucekkine, R., De La Croix, D., & Licandro, O. (2011). Vintage capital growth theory: Three breakthroughs. *Economic Growth and Development*, *11*, 87–116.

Breeden, D. T. (1979). An intertemporal asset pricing model with stochastic consumption and investment opportunities. In *Theory of valuation* (pp. 53–96). World Scientific.

Broda, C. & Weinstein, D. E. (2010). Product creation and destruction: Evidence and price implications. *The American Economic Review*, *100*(3), 691–723.

Carvalho, V. M. & Gabaix, X. (2013). The great diversification and its undoing. *American Economic Review*, *103*(5), 1697–1727.

Carvalho, V. M. & Grassi, B. (2015). Large firm dynamics and the business cycle.

Chari, V. V., Christiano, L., & Kehoe, P. (2007). The gertler-gilchrist evidence on small and large firm sales. *manuscript, January*.

Clementi, G. L. & Palazzo, B. (2016). Entry, exit, firm dynamics, and aggregate fluctuations. *American Economic Journal: Macroeconomics*, *8*(3), 1–41.

Clementi, G. L. & Palazzo, B. (2018). Investment and the cross-section of equity returns. *The Journal of Finance*, *0*(0).

Comin, D. & Mulani, S. (2004). *Diverging trends in macro and micro volatility: Facts*. National Bureau of Economic Research.

Covas, F. & den Haan, W. J. (2011). The cyclical behavior of debt and equity finance. *American Economic Review*, *101*(2), 877–899.

Crouzet, N. & Mehrotra, N. R. (2017). Small and large firms over the business cycle. *Unpublished*.

Davydenko, S. A. (2012). When do firms default? a study of the default boundary.

De Loecker, J., Eeckhout, J., & Unger, G. (2017). *The rise of market power and the macroeconomic implications*. National Bureau of Economic Research.

Decker, R. A., D'Erasmo, P. N., & Moscoso Boedo, H. (2016). Market exposure and endogenous firm volatility over the business cycle. *American Economic Journal: Macroeconomics*, *8*(1), 148–98.

Dixit & Stiglitz. (1977). Monopolistic competition and optimum product diversity. *The American Economic Review*, *67*(3), 297–308.

Donangelo, A., Gourio, F., Kehrig, M., & Palacios, M. (2017). The cross-section of labor leverage and equity returns.

Doob, J. L. (1953). *Stochastic processes*. Wiley New York.

Dunne, T. (1991). Technology usage in us manufacturing industries: New evidence from the survey of manufacturing technology. *91*(7).

Dutta, J. & Polemarchakis, H. (1992). *On the aggregate effects of individual risks*. Université catholique de Louvain, Center for Operations Research and Econometrics (CORE).

Fama, E. F. & French, K. R. (1992). The cross-section of expected stock returns. *the Journal of Finance*, *47*(2), 427–465.

Fama, E. F. & French, K. R. (2008). Dissecting anomalies. *The Journal of Finance*, *63*(4), 1653–1678.

Fama, E. F. & French, K. R. (2016). Dissecting anomalies with a five-factor model. *The Review of Financial Studies*, *29*(1), 69–103.

Feldman, M. & Gilles, C. (1985). An expository note on individual risk without aggregate uncertainty. *Journal of Economic Theory*, *35*(1), 26–32.

Felipe, J. & Fisher, F. M. (2003). Aggregation in production functions: What applied economists should know. *Metroeconomica*, *54*(2-3), 208–262.

Forbes, K. J. (2012). The "big c": Identifying and mitigating contagion.

Foster, L., Haltiwanger, J. C., & Krizan, C. J. (2001). Aggregate productivity growth: Lessons from microeconomic evidence. In *New developments in productivity analysis* (pp. 303–372). University of Chicago Press.

Frankel, M. (1955). Obsolescence and technological change in a maturing economy. *The American Economic Review*, *45*(3), 296–319.

Fudenberg, D., Gilbert, R., Stiglitz, J., & Tirole, J. (1983). Preemption, leapfrogging and competition in patent races. *European Economic Review*, *22*(1), 3–31.

Gabaix, X. (2011). The granular origins of aggregate fluctuations. *Econometrica*, *79*(3), 733–772.

Gertler, M. & Gilchrist, S. (1994). Monetary policy, business cycles, and the behavior of small manufacturing firms. *The Quarterly Journal of Economics*, *109*(2), 309–340.

Ghironi, F. & Melitz, M. J. (2005). International trade and macroeconomic dynamics with heterogeneous firms. *The Quarterly Journal of Economics*, *120*(3), 865–915.

Gollop, F. M. & Monahan, J. L. (1991). A generalized index of diversification: Trends in u.s. manufacturing. *The Review of Economics and Statistics*, *73*(2), 318.

Gomes, J. F. (2001). Financing investment. *American Economic Review*, 1263–1285.

Gomes, J. F., Kogan, L., & Zhang, L. (2003). Equilibrium cross section of returns. *Journal of Political Economy*, *111*(4), 693–732.

Gorman, W. M. (1959). Separable utility and aggregation. *Econometrica: Journal of the Econometric Society*, 469–481.

Gort, M. (1962). The directions of diversification. In *Diversification and integration in american industry* (pp. 103–134). National Bureau of Economic Research, Inc.

Gourio, F. (2007). Labor leverage, firms heterogeneous sensitivities to the business cycle, and the cross-section of expected returns. *Unpublished working paper, Boston University.*

Hausman, J. (2001). Mismeasured variables in econometric analysis: Problems from the right and problems from the left. *Journal of Economic perspectives*, *15*(4), 57–67.

Herskovic, B., Kelly, B., Lustig, H., & Van Nieuwerburgh, S. (2017). *Firm volatility in granular networks*. Chicago Booth Research Paper No. 12-56, Fama-Miller Working Paper.

Hopenhayn, H. A. (1992). Entry, exit, and firm dynamics in long run equilibrium. *Econometrica*, *60*(5), 1127.

Houthakker, H. S. (1955). The pareto distribution and the cobb-douglas production function in activity analysis. *The Review of Economic Studies*, *23*(1), 27.

Hulten, C. R. (1978). Growth accounting with intermediate inputs. *The Review of Economic Studies*, *45*(3), 511–518.

İmrohoroğlu, A. & Tuzel, Ş. (2014). Firm-level productivity, risk, and return. *Management Science*, *60*(8), 2073–2090.

Jermann, U. J. (1998). Asset pricing in production economies. *Journal of monetary Economics*, *41*(2), 257–275.

Johansen, L. (1959). Substitution versus fixed production coefficients in the theory of economic growth: A synthesis. *Econometrica: Journal of the Econometric Society*, 157–176.

Jovanovic, B. (1982). Selection and the evolution of industry. *Econometrica: Journal of the Econometric Society*, 649–670.

Judd, K. L. (1985). The law of large numbers with a continuum of iid random variables. *Journal of Economic Theory*, *35*(1), 19–25.

Khan, M. A. & Sun, Y. (1999). Weak measurability and characterizations of risk. *Economic Theory*, *13*(3), 541–560.

Lettau, M. (2003). Inspecting the mechanism: Closed-form solutions for asset prices in real business cycle models. *Economic Journal*, *113*(489), 550–575.

Lewis, J. B. & Linzer, D. A. (2005). Estimating regression models in which the dependent variable is based on estimates. *Political analysis*, *13*(4), 345–364.

Lintner, J. (1965). The valuation of risk assets and the selection of risky investments in stock portfolios and capital budgets. In *Stochastic optimization models in finance* (pp. 131–155). Elsevier.

Malinvaud, E. (1972). The allocation of individual risks in large markets. *Journal of Economic Theory*, 312–328.

Malinvaud, E. (1973). Markets for an exchange economy with individual risks. *Econometrica*, *41*(3), 383–410.

Markowitz, H. (1952). Portfolio selection. *The journal of finance*, *7*(1), 77–91.

Martin, J. D. & Sayrak, A. (2003). Corporate diversification and shareholder value: A survey of recent literature. *Journal of corporate finance*, *9*(1), 37–57.

Melitz, M. J. (2003). The impact of trade on intra-industry reallocations and aggregate industry productivity. *Econometrica*, *71*(6), 1695–1725.

Milgrom, P. & Roberts, J. (1990). The economics of modern manufacturing: Technology, strategy, and organization. *The American Economic Review*, *80*(3), 511–528.

Moscarini, G. & Postel-Vinay, F. (2009). *Large employers are more cyclically sensitive*. National Bureau of Economic Research.

Mossin, J. (1966). Equilibrium in a capital asset market. *Econometrica: Journal of the econometric society*, 768–783.

Al-Najjar, N. (1995). Decomposition and characterization of risk with a continuum of random variables. *Econometrica: Journal of the Econometric Society*, 1195–1224.

Olley, G. S. & Pakes, A. (1996). The dynamics of productivity in the telecommunications equipment industry. *Econometrica*, *64*(6), 1263–1297.

Rajan, R. G. & Zingales, L. (1995). What do we know about capital structure? some evidence from international data. *The journal of Finance*, *50*(5), 1421–1460.

Reinganum, J. F. (1981). Dynamic games of innovation. *Journal of Economic Theory*, *25*(1), 21–41.

Romer, P. M. (2016). The trouble with macroeconomics. *Unpublished*.

Rouwenhorst, G. (1995). Asset pricing implications of equilibrium business cycle models. (Chap. 10).

Sharpe, W. F. (1964). Capital asset prices: A theory of market equilibrium under conditions of risk. *The journal of finance*, *19*(3), 425–442.

Solow, R. M. (1960). Investment and technical progress. *Mathematical methods in the social sciences*, *1*, 48–93.

Stiglitz. (2011). Rethinking macroeconomics: What failed, and how to repair it. *Journal of the European Economic Association*, *9*(4), 591–645.

van Dijk, M. A. (2011). Is size dead? a review of the size effect in equity returns. *Journal of Banking & Finance*, *35*(12), 3263–3274.

Villalonga, B. (2004). Diversification discount or premium? new evidence from the business information tracking series. *The Journal of Finance*, *59*(2), 479–506.

Wang, D. (2012). Do diversified and focused firms have different growth options? evidence from total asset growth.

White, H. (1980). A heteroskedasticity-consistent covariance matrix estimator and a direct test for heteroskedasticity. *Econometrica: Journal of the Econometric Society*, 817–838.

Zhang, L. (2017). *The investment CAPM*. National Bureau of Economic Research.